7. Power Spectrum Analysis

Definition: Fourier Transform

Functions $g(t)$ and $G(f)$ are a Fourier transform pair if

$$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt, \qquad g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df$$

Definition: Power Spectral Density

The power spectral density function of the WSS stochastic process $X(t)$ is

$$S_X(f)=\lim_{T\to\infty}\frac{1}{2T}E\left[\left|X_T(f)\right|^2\right]$$

$$=\lim_{T\to\infty}\frac{1}{2T}E\left[\left|\int_{-T}^{T} X(t)e^{-j2\pi ft}dt\right|^2\right].$$

Theorem 7.1

If $X(t)$ is a WSS stochastic process, $R_X(\tau)$ and $S_X(f)$ are the Fourier transform pair

$$S_X(f)=\int_{-\infty}^{\infty}R_X(\tau)e^{-j2\pi f\tau}d\tau,\qquad R_X(\tau)=\int_{-\infty}^{\infty}S_X(f)e^{j2\pi f\tau}df$$

Pf :

Theorem 7.2

For a WSS stochastic process $X(t)$, the power spectral density $S_X(f)$ is a real-value function with the following properties:

(a) $S_X(f) \geq 0$ for all f

(b) $\int_{-\infty}^{\infty} S_X(f)\,df = E\left[X^2(t)\right] = R_X(0)$

(c) $S_X(-f) = S_X(f)$

Pf :

Example 7.1

A WSS stochastic process $X(t)$ has autocorrelation function $R_X(\tau) = Ae^{-b|\tau|}$ where $b > 0$. Derive the power spectral density function $S_X(f)$ and calculate the average power $E[X^2(t)]$.

Sol :

$$Ans: S_X(f) = \frac{2Ab}{b^2 + (2\pi f)^2}, E[X^2(t)] = A$$

Example 7.2

A white Gaussian noise process $X(t)$ with autocorrelation function $R_W(\tau) = \eta_0 \delta(\tau)$ is passed through the moving-average filter

$$h(t) = \begin{cases} 1/T & 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases}$$

For the output $Y(t)$, find the power spectral density $S_Y(f)$.

Sol :

$$Ans: S_Y(f) = \frac{2\eta_0 [1 - \cos(2\pi f T)]}{(2\pi f T)^2}$$

Discrete-Time Fourier Transform (DTFT)

Definition :

The sequence $\{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\}$ and the function $X(\phi)$ are a discrete-time Fourier transform (DTFT) pair if

$$X(\phi) = \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi\phi n}, \qquad x_n = \int_{-1/2}^{1/2} X(\phi) e^{j2\pi\phi n} d\phi$$

Example 7.3 : Calculate the DTFT $H(\phi)$ of the order $M-1$ moving-average filter h_n of Example 6.6.

$$h_n = \begin{cases} 1/M & n = 0, \cdots, M-1 \\ 0 & \text{otherwise.} \end{cases}$$

Sol :

$$Ans : H(\phi) = \frac{1}{M}\left(\frac{1-e^{-j2\pi\phi M}}{1-e^{-j2\pi\phi}}\right)$$

Power Spectral Density of a Random Sequence

Definition :

The power spectral density function of the WSS random sequence X_n is

$$S_X(\phi) = \lim_{L\to\infty} \frac{1}{2L+1} E\left[\left|\sum_{n=-L}^{L} X_n e^{-j2\pi\phi n}\right|^2\right].$$

Theorem 7.3 : Discrete-Time Winer-Khintchine

If X_n is a WSS stochastic process, $R_X[k]$ and $S_X(\phi)$ are a discrete-time Fourier transform pair :

$$S_X(\phi) = \sum_{k=-\infty}^{\infty} R_X[k] e^{-j2\pi\phi k}, \quad R_X[k] = \int_{-1/2}^{1/2} S_X(\phi) e^{j2\pi\phi k} d\phi$$

Theorem 7.4

For a WSS random sequence X_n, the power spectral density $S_X(\phi)$ has the following properties:

(a) $S_X(\phi) \geq 0$ for all ϕ,

(b) $\int_{-1/2}^{1/2} S_X(\phi)\, d\phi = E\left[X_n^{\,2}\right] = R_X[0]$.

(c) $S_X(-\phi) = S_X(\phi)$,

(d) for any integer n, $S_X(\phi + n) = S_X(\phi)$.

Example 7.4

The WSS random sequence X_n has zero expected value and autocorrelation function $R_X[k]$ as follows. Derive the power spectral density function of X_n.

$$R_X[k] = \begin{cases} \sigma^2 (2 - |n|)/4 & n = -1,0,1, \\ 0 & \text{otherwise.} \end{cases}$$

Sol :

$$Ans : S_X(\phi) = \frac{\sigma^2}{2}\left(1 + \cos(2\pi\phi)\right)$$

Example 7.5

The WSS random sequence X_n has zero mean and power spectral density $S_X(\phi)$ as follows. What is the autocorrelation function $R_X[k]$.

$$S_X(\phi) = \frac{1}{2}\delta(\phi - \phi_0) + \frac{1}{2}\delta(\phi + \phi_0), \text{ where } 0 < \phi_0 < 1/2.$$

Sol :

$$Ans: R_X[k] = \cos(2\pi\phi_0 k)$$

Assignment

Q.1.

Let $Y(t) = X(t) + N(t)$ where $N(t)$ is a WSS noise process with $\mu_N = 0.$

In this case, when $X(t)$ and $N(t)$ are jointly WSS, we found that

$$R_Y(\tau) = R_X(\tau) + R_{XN}(\tau) + R_{NX}(\tau) + R_N(\tau).$$

Find the power spectral density of the output $Y(t)$.