# Exponential random variable

**Definition:** A continuous RV is called exponential with parameter $\lambda > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

**Features of** $f(x)$

- Properly normalized:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_0^{\infty} = (0+1) = 1$$

**Cumulative distribution function:**

- Evaluation:

$$\begin{aligned} F(t) &= P(X \leq t) \\ &= \int_{-\infty}^{t} f(x)dx = \begin{cases} 1 - e^{-\lambda t}, & t > 0 \\ 0, & t \leq 0 \end{cases} \end{aligned}$$

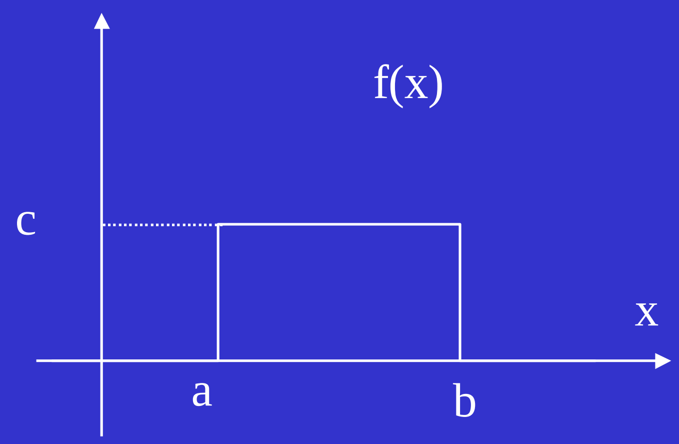
# Uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & x > b \end{cases}$$

$$\mu = \frac{1}{2}(a+b)$$

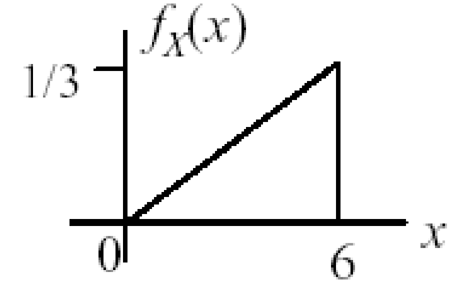
$$\sigma^2 = \frac{1}{12}(b-a)^2$$



**Example:**

Now consider a random variable $X$ with a probability density function:

$$f_X(x) = \begin{cases} \frac{1}{18}x & 0 \le x \le 6 \\ 0 & \text{otherwise.} \end{cases}$$




$$m_X = E[X] = \frac{1}{18}\int_0^6 x^2\,dx = \left.\frac{x^3}{54}\right|_0^6 = 4$$

$$\sigma_X^2 = E\left[(X - m_X)^2\right] = \frac{1}{18}\int_0^6 (x-4)^2\,x\,dx$$

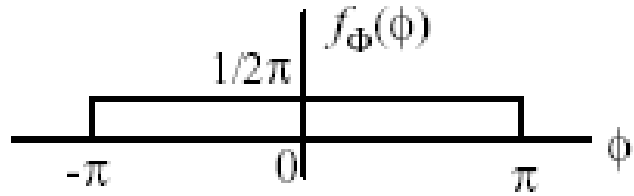
$$= \frac{1}{18}\int_0^6 \left(x^3 - 8x^2 + 16x\right)dx = \frac{1}{18}\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^6 = 2.$$

**Example:**

$X = A\cos(\omega t_0 + \phi)$ $\phi$ is a uniform random variable: $[-\pi, \pi]$

$X = g(\phi)$

$E[X] = E_\phi[g(\phi)] = \int g(\phi) f_\Phi(\phi)\, d\phi$




(a) Mean

$$
\begin{aligned}
m_X &= E\left[A\cos\left(\omega t_0 + \phi\right)\right] \\
&= A\int_{-\pi}^{\pi} \cos\left(\omega t_0 + \phi\right)\frac{1}{2\pi} d\phi \\
&= \frac{A}{2\pi}\cdot \sin\left(\omega t_0 + \phi\right)\Big|_{-\pi}^{\pi} \\
&= \frac{A}{2\pi}\left[\sin\left(\omega t_0 + \pi\right) - \sin\left(\omega t_0 - \pi\right)\right] \\
&= \frac{A}{2\pi}\left[\sin \omega t_0 \cos\pi + \cos\omega t_0 \underbrace{\sin\pi}_{0} - \sin\omega t_0 \cos\pi + \cos\omega t_0 \underbrace{\sin\pi}_{0}\right] = 0
\end{aligned}
$$

(b) Variance

$$\sigma_X^2 = E\left[ A^2 \cos^2(\omega t_0 + \phi) \right]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos^2(\omega t_0 + \phi)\, d\phi = \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1+\cos(2\omega t_0 + 2\phi)}{2}\, d\phi$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} d\phi + \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \cos(2\omega t_0 + 2\phi)\, d\phi$$

$$= \frac{A^2}{4\pi}\cdot \phi\Big|_{-\pi}^{\pi} + \frac{A^2}{4\pi}\frac{\sin(2\omega t_0 + 2\phi)}{2}\Big|_{-\pi}^{\pi}$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin(2\omega t_0 + 2\pi) - \sin(2\omega t_0 - 2\pi)\right]$$

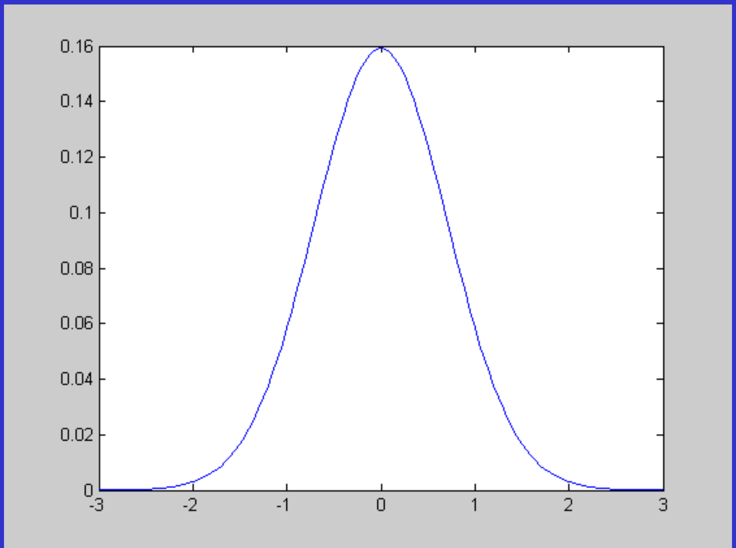
$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin 2\omega t_0 - \sin 2\omega t_0\right] = \frac{A^2}{2}$$

# Gaussian random variable

$$\forall x \qquad f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}}$$

$$E(T)=m \qquad V(X)=\sigma^2$$

*If* $\quad X \approx N(m;\sigma) \Rightarrow \frac{X-m}{\sigma} \approx N(0;1)$



## Evaluating Probabilities for Gaussian Random Variables

Relations:

$$F_X(x) = \Pr[X \le x] = \Phi\left(\frac{x-m}{\sigma}\right) = 1 - Q\left(\frac{x-m}{\sigma}\right)$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz \qquad Q(y) = 1 - \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-\frac{z^2}{2}} dz$$

Due to symmetry of the Gaussian density:

$$\Phi(-y) = Q(y) \qquad Q(-z) = 1 - Q(z).$$

These functions are tabulated.

Approximation: when $y \ge 2$, the Q function can be written as

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-z^2/2} dz \approx \frac{1}{\sqrt{2\pi}\, y} e^{-y^2/2}$$

**Example**:

A random variable $X$ is Gaussian with a mean value of $m$ and a standard deviation of $\sigma$.

a) Find $\Pr[m-2\sigma < X \leq m+2\sigma]$.

$$\Pr[m-2\sigma < X \leq m+2\sigma] = 1-\Pr[X \leq m-2\sigma]-\Pr[X > m+2\sigma]$$

$$= 1-2\Pr[X > m+2\sigma] = 1-2Q\left(\frac{m+2\sigma-m}{\sigma}\right)$$

$$= 1-2Q(2) = 1-4.56\times10^{-2} = 0.9544.$$

b) Given that mean $m$ = 5 and variance $\sigma^2$ = 64, determine the probability $\Pr[m - \sigma < X \leq m + \sigma]$.

Since $m$ = 5 and $\sigma$ = 8, we need to determine:

$$\Pr[-3 < X \leq 13] = 1-\Pr[X \leq -3]-\Pr[X > 13] = 1-2\Pr[X > 13]$$

$$= 1-2Q\left(\frac{13-5}{8}\right) = 1-2Q(1) = 1-2\times0.159 = 0.682.$$

**Table of Q function values**

| $x$ | $Q(x)$ | $x$ | $Q(x)$ | $x$ | $Q(x)$ | $x$ | $Q(x)$ |
|---|---|---|---|---|---|---|---|
| 0.00 | 5.0000E-01 | 1.00 | 1.5866E-01 | 2.00 | 2.2750E-02 | 5.00 | 2.8665E-07 |
| 0.02 | 4.9202E-01 | 1.02 | 1.5386E-01 | 2.05 | 2.0182E-02 | 5.10 | 1.6983E-07 |
| 0.04 | 4.8405E-01 | 1.04 | 1.4917E-01 | 2.10 | 1.7864E-02 | 5.20 | 9.9644E-08 |
| 0.06 | 4.7608E-01 | 1.06 | 1.4457E-01 | 2.15 | 1.5778E-02 | 5.30 | 5.7901E-08 |
| 0.08 | 4.6812E-01 | 1.08 | 1.4007E-01 | 2.20 | 1.3903E-02 | 5.40 | 3.3320E-08 |
| 0.10 | 4.6017E-01 | 1.10 | 1.3567E-01 | 2.25 | 1.2224E-02 | 5.50 | 1.8990E-08 |
| 0.12 | 4.5224E-01 | 1.12 | 1.3136E-01 | 2.30 | 1.0724E-02 | 5.60 | 1.0718E-08 |
| 0.14 | 4.4433E-01 | 1.14 | 1.2714E-01 | 2.35 | 9.3867E-03 | 5.70 | 5.9904E-09 |
| 0.16 | 4.3644E-01 | 1.16 | 1.2302E-01 | 2.40 | 8.1975E-03 | 5.80 | 3.3157E-09 |
| 0.18 | 4.2858E-01 | 1.18 | 1.1900E-01 | 2.45 | 7.1428E-03 | 5.90 | 1.8175E-09 |
| 0.20 | 4.2074E-01 | 1.20 | 1.1507E-01 | 2.50 | 6.2097E-03 | 6.00 | 9.8659E-10 |
| 0.22 | 4.1294E-01 | 1.22 | 1.1123E-01 | 2.55 | 5.3861E-03 | 6.10 | 5.3034E-10 |
| 0.24 | 4.0517E-01 | 1.24 | 1.0749E-01 | 2.60 | 4.6612E-03 | 6.20 | 2.8232E-10 |
| 0.26 | 3.9743E-01 | 1.26 | 1.0383E-01 | 2.65 | 4.0246E-03 | 6.30 | 1.4882E-10 |
| 0.28 | 3.8974E-01 | 1.28 | 1.0027E-01 | 2.70 | 3.4670E-03 | 6.40 | 7.7688E-11 |
| 0.30 | 3.8209E-01 | 1.30 | 9.6800E-02 | 2.75 | 2.9798E-03 | 6.50 | 4.0160E-11 |
| 0.32 | 3.7448E-01 | 1.32 | 9.3418E-02 | 2.80 | 2.5551E-03 | 6.60 | 2.0558E-11 |
| 0.34 | 3.6693E-01 | 1.34 | 9.0123E-02 | 2.85 | 2.1860E-03 | 6.70 | 1.0421E-11 |
| 0.36 | 3.5942E-01 | 1.36 | 8.6915E-02 | 2.90 | 1.8658E-03 | 6.80 | 5.2310E-12 |
| 0.38 | 3.5197E-01 | 1.38 | 8.3793E-02 | 2.95 | 1.5889E-03 | 6.90 | 2.6001E-12 |
| 0.40 | 3.4458E-01 | 1.40 | 8.0757E-02 | 3.00 | 1.3499E-03 | 7.00 | 1.2798E-12 |
| 0.42 | 3.3724E-01 | 1.42 | 7.7804E-02 | 3.05 | 1.1442E-03 | 7.10 | 6.2378E-13 |
| 0.44 | 3.2997E-01 | 1.44 | 7.4934E-02 | 3.10 | 9.6760E-04 | 7.20 | 3.0106E-13 |
| 0.46 | 3.2276E-01 | 1.46 | 7.2145E-02 | 3.15 | 8.1635E-04 | 7.30 | 1.4388E-13 |
| 0.48 | 3.1561E-01 | 1.48 | 6.9437E-02 | 3.20 | 6.8714E-04 | 7.40 | 6.8092E-14 |
| 0.50 | 3.0854E-01 | 1.50 | 6.6807E-02 | 3.25 | 5.7703E-04 | 7.50 | 3.1909E-14 |
| 0.52 | 3.0153E-01 | 1.52 | 6.4255E-02 | 3.30 | 4.8342E-04 | 7.60 | 1.4807E-14 |
| 0.54 | 2.9460E-01 | 1.54 | 6.1780E-02 | 3.35 | 4.0406E-04 | 7.70 | 6.8033E-15 |
| 0.56 | 2.8774E-01 | 1.56 | 5.9380E-02 | 3.40 | 3.3693E-04 | 7.80 | 3.0954E-15 |
| 0.58 | 2.8096E-01 | 1.58 | 5.7053E-02 | 3.45 | 2.8029E-04 | 7.90 | 1.3945E-15 |
| 0.60 | 2.7425E-01 | 1.60 | 5.4799E-02 | 3.50 | 2.3263E-04 | 8.00 | 6.2210E-16 |
| 0.62 | 2.6763E-01 | 1.62 | 5.2616E-02 | 3.55 | 1.9262E-04 | 8.10 | 2.7480E-16 |
| 0.64 | 2.6109E-01 | 1.64 | 5.0503E-02 | 3.60 | 1.5911E-04 | 8.20 | 1.2019E-16 |
| 0.66 | 2.5463E-01 | 1.66 | 4.8457E-02 | 3.65 | 1.3112E-04 | 8.30 | 5.2056E-17 |
| 0.68 | 2.4825E-01 | 1.68 | 4.6479E-02 | 3.70 | 1.0780E-04 | 8.40 | 2.2324E-17 |
| 0.70 | 2.4196E-01 | 1.70 | 4.4565E-02 | 3.75 | 8.8417E-05 | 8.50 | 9.4795E-18 |
| 0.72 | 2.3576E-01 | 1.72 | 4.2716E-02 | 3.80 | 7.2348E-05 | 8.60 | 3.9858E-18 |
| 0.74 | 2.2965E-01 | 1.74 | 4.0930E-02 | 3.85 | 5.9059E-05 | 8.70 | 1.6594E-18 |
| 0.76 | 2.2363E-01 | 1.76 | 3.9204E-02 | 3.90 | 4.8096E-05 | 8.80 | 6.8408E-19 |
| 0.78 | 2.1770E-01 | 1.78 | 3.7538E-02 | 3.95 | 3.9076E-05 | 8.90 | 2.7923E-19 |
| 0.80 | 2.1186E-01 | 1.80 | 3.5930E-02 | 4.00 | 3.1671E-05 | 9.00 | 1.1286E-19 |
| 0.82 | 2.0611E-01 | 1.82 | 3.4380E-02 | 4.10 | 2.0658E-05 | 9.10 | 4.5166E-20 |
| 0.84 | 2.0045E-01 | 1.84 | 3.2884E-02 | 4.20 | 1.3346E-05 | 9.20 | 1.7897E-20 |
| 0.86 | 1.9489E-01 | 1.86 | 3.1443E-02 | 4.30 | 8.5399E-06 | 9.30 | 7.0223E-21 |
| 0.88 | 1.8943E-01 | 1.88 | 3.0054E-02 | 4.40 | 5.4125E-06 | 9.40 | 2.7282E-21 |
| 0.90 | 1.8406E-01 | 1.90 | 2.8717E-02 | 4.50 | 3.3977E-06 | 9.50 | 1.0495E-21 |
| 0.92 | 1.7879E-01 | 1.92 | 2.7429E-02 | 4.60 | 2.1125E-06 | 9.60 | 3.9972E-22 |
| 0.94 | 1.7361E-01 | 1.94 | 2.6190E-02 | 4.70 | 1.3008E-06 | 9.70 | 1.5075E-22 |
| 0.96 | 1.6853E-01 | 1.96 | 2.4998E-02 | 4.80 | 7.9333E-07 | 9.80 | 5.6293E-23 |
| 0.98 | 1.6354E-01 | 1.98 | 2.3852E-02 | 4.90 | 4.7918E-07 | 9.90 | 2.0814E-23 |

**Table of inverse Q function values**

| $l$ | $x=Q^{-1}(10^{-l})$ | $l$ | $x=Q^{-1}(10^{-l})$ | $l$ | $x=Q^{-1}(10^{-l})$ | $l$ | $x=Q^{-1}(10^{-l})$ |
|---|---|---|---|---|---|---|---|
| 1 | 1.2816 | 7 | 5.1993 | 13 | 7.3488 | 19 | 9.0133 |
| 2 | 2.3263 | 8 | 5.6120 | 14 | 7.6506 | 20 | 9.2623 |
| 3 | 3.0902 | 9 | 5.9978 | 15 | 7.9413 | 21 | 9.5050 |
| 4 | 3.7190 | 10 | 6.3613 | 16 | 8.2221 | 22 | 9.7418 |
| 5 | 4.2649 | 11 | 6.7060 | 17 | 8.4938 | 23 | 9.9730 |
| 6 | 4.7534 | 12 | 7.0345 | 18 | 8.7573 | 24 | 10.1990 |

**Table of erf function values**

| $x$ | erf( $x$ ) | $x$ | erf( $x$ ) | $x$ | erf( $x$ ) | $x$ | erf( $x$ ) |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.00000 | 0.40 | 0.42839 | 0.80 | 0.74210 | 1.20 | 0.91031 |
| 0.01 | 0.01128 | 0.41 | 0.43797 | 0.81 | 0.74800 | 1.21 | 0.91296 |
| 0.02 | 0.02257 | 0.42 | 0.44747 | 0.82 | 0.75381 | 1.22 | 0.91553 |
| 0.03 | 0.03384 | 0.43 | 0.45689 | 0.83 | 0.75952 | 1.23 | 0.91805 |
| 0.04 | 0.04511 | 0.44 | 0.46623 | 0.84 | 0.76514 | 1.24 | 0.92051 |
| 0.05 | 0.05637 | 0.45 | 0.47548 | 0.85 | 0.77067 | 1.25 | 0.92290 |
| 0.06 | 0.06762 | 0.46 | 0.48466 | 0.86 | 0.77610 | 1.30 | 0.93401 |
| 0.07 | 0.07886 | 0.47 | 0.49375 | 0.87 | 0.78144 | 1.35 | 0.94376 |
| 0.08 | 0.09008 | 0.48 | 0.50275 | 0.88 | 0.78669 | 1.40 | 0.95229 |
| 0.09 | 0.10128 | 0.49 | 0.51167 | 0.89 | 0.79184 | 1.45 | 0.95970 |
| 0.10 | 0.11246 | 0.50 | 0.52050 | 0.90 | 0.79691 | 1.50 | 0.96611 |
| 0.11 | 0.12362 | 0.51 | 0.52924 | 0.91 | 0.80188 | 1.55 | 0.97162 |
| 0.12 | 0.13476 | 0.52 | 0.53790 | 0.92 | 0.80677 | 1.60 | 0.97635 |
| 0.13 | 0.14587 | 0.53 | 0.54646 | 0.93 | 0.81156 | 1.65 | 0.98038 |
| 0.14 | 0.15695 | 0.54 | 0.55494 | 0.94 | 0.81627 | 1.70 | 0.98379 |
| 0.15 | 0.16800 | 0.55 | 0.56332 | 0.95 | 0.82089 | 1.75 | 0.98667 |
| 0.16 | 0.17901 | 0.56 | 0.57162 | 0.96 | 0.82542 | 1.80 | 0.98909 |
| 0.17 | 0.18999 | 0.57 | 0.57982 | 0.97 | 0.82987 | 1.85 | 0.99111 |
| 0.18 | 0.20094 | 0.58 | 0.58792 | 0.98 | 0.83423 | 1.90 | 0.99279 |
| 0.19 | 0.21184 | 0.59 | 0.59594 | 0.99 | 0.83851 | 1.95 | 0.99418 |
| 0.20 | 0.22270 | 0.60 | 0.60386 | 1.00 | 0.84270 | 2.00 | 0.99532 |
| 0.21 | 0.23352 | 0.61 | 0.61168 | 1.01 | 0.84681 | 2.05 | 0.99626 |
| 0.22 | 0.24430 | 0.62 | 0.61941 | 1.02 | 0.85084 | 2.10 | 0.99702 |
| 0.23 | 0.25502 | 0.63 | 0.62705 | 1.03 | 0.85478 | 2.15 | 0.99764 |
| 0.24 | 0.26570 | 0.64 | 0.63459 | 1.04 | 0.85865 | 2.20 | 0.99814 |
| 0.25 | 0.27633 | 0.65 | 0.64203 | 1.05 | 0.86244 | 2.25 | 0.99854 |
| 0.26 | 0.28690 | 0.66 | 0.64938 | 1.06 | 0.86614 | 2.30 | 0.99886 |
| 0.27 | 0.29742 | 0.67 | 0.65663 | 1.07 | 0.86977 | 2.35 | 0.99911 |
| 0.28 | 0.30788 | 0.68 | 0.66378 | 1.08 | 0.87333 | 2.40 | 0.99931 |
| 0.29 | 0.31828 | 0.69 | 0.67084 | 1.09 | 0.87680 | 2.45 | 0.99947 |
| 0.30 | 0.32863 | 0.70 | 0.67780 | 1.10 | 0.88021 | 2.50 | 0.99959 |
| 0.31 | 0.33891 | 0.71 | 0.68467 | 1.11 | 0.88353 | 2.55 | 0.99969 |
| 0.32 | 0.34913 | 0.72 | 0.69143 | 1.12 | 0.88679 | 2.60 | 0.99976 |
| 0.33 | 0.35928 | 0.73 | 0.69810 | 1.13 | 0.88997 | 2.65 | 0.99982 |
| 0.34 | 0.36936 | 0.74 | 0.70468 | 1.14 | 0.89308 | 2.70 | 0.99987 |
| 0.35 | 0.37938 | 0.75 | 0.71116 | 1.15 | 0.89612 | 2.75 | 0.99990 |
| 0.36 | 0.38933 | 0.76 | 0.71754 | 1.16 | 0.89910 | 2.80 | 0.99992 |
| 0.37 | 0.39921 | 0.77 | 0.72382 | 1.17 | 0.90200 | 2.85 | 0.99994 |
| 0.38 | 0.40901 | 0.78 | 0.73001 | 1.18 | 0.90484 | 2.90 | 0.99996 |
| 0.39 | 0.41874 | 0.79 | 0.73610 | 1.19 | 0.90761 | 2.95 | 0.99997 |

## Evaluating Probabilities for Gaussian Random Variables Using the "Error Function"

Relations:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-m)^2}{2\sigma^2}} dz = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x-m}{\sqrt{2}\,\sigma}\right)\right)$$

where

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-z^2} dz \qquad \text{and} \qquad \text{erf}(-y) = -\text{erf}(y)$$

A "complementary error function" is sometimes also used:

$$\text{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^{\infty} e^{-z^2} dz = \frac{1}{2} - \text{erf}(y)$$

The "complementary error function" is related to the Q function:

$$Q(y) = \frac{1}{2}\,\text{erfc}\left(\frac{y}{\sqrt{2}}\right)$$

Both "erf" and "erfc" are available in MATLAB.

**Example:**

A random variable $X$ is Gaussian with a mean value of $m=1$ and variance $\sigma^2 = 6.25$.

Find (a) $\Pr[X \le 1.7]$ (b) $\Pr[X \le 0.2]$ (c) $\Pr[X > 3.5]$

$$\text{(a)} \quad \Pr[X \le 1.7] = \int_{-\infty}^{1.7} f_X(x)dx = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{1.7-1}{2.5\sqrt{2}}\right)\right] = 0.6103$$

$$\text{(b)} \quad \Pr[X \le 0.2] = \int_{-\infty}^{0.2} f_X(x)dx = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{0.2-1}{2.5\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left[1-\operatorname{erf}\left(\frac{1-0.2}{2.5\sqrt{2}}\right)\right] = 0.3745$$

$$\text{(c)} \quad \Pr[X > 3.5] = 1-\Pr[X \le 3.5] = 1-\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{3.5-1}{2.5\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left[1-\operatorname{erf}\left(\frac{3.5-1}{2.5\sqrt{2}}\right)\right] = 0.1587$$

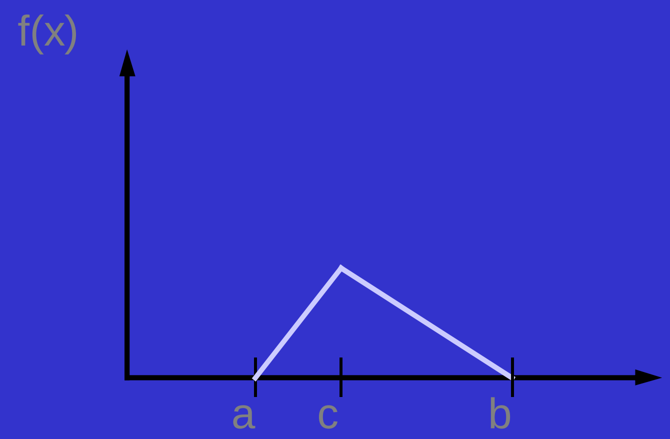
# Triangular random variable

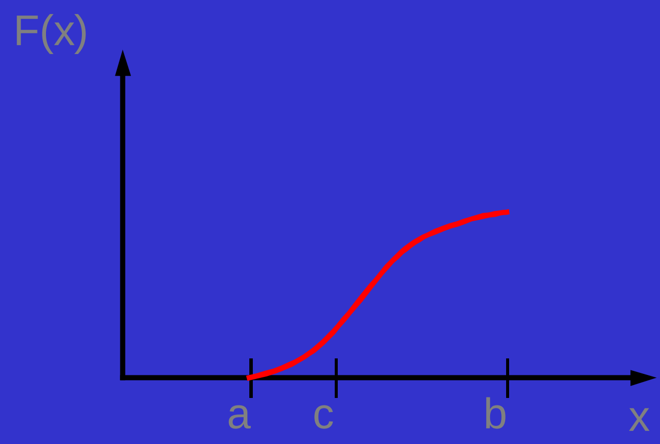
$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)} & \text{if } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{if } c \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{si } x < a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{si } a \le x \le c \\ 1-\dfrac{(b-x)^2}{(b-a)(b-c)} & \text{if } c \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

$$\mu = \frac{1}{3}(a+b+c)$$

$$\sigma^2 = \frac{a^2+b^2+c^2-ab-ac-bc}{18}$$



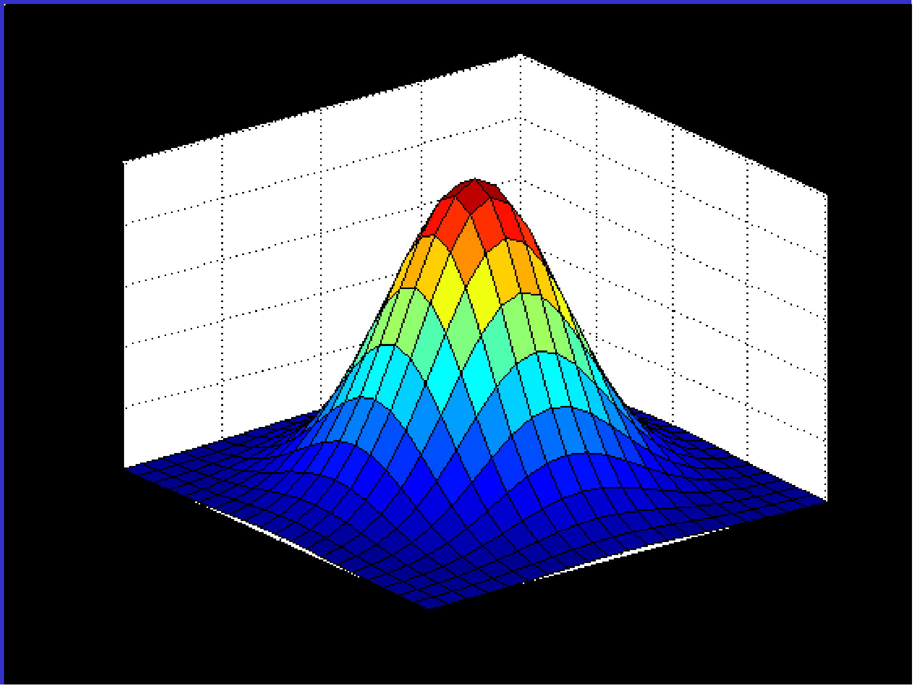



# Bi-dimensional random variable

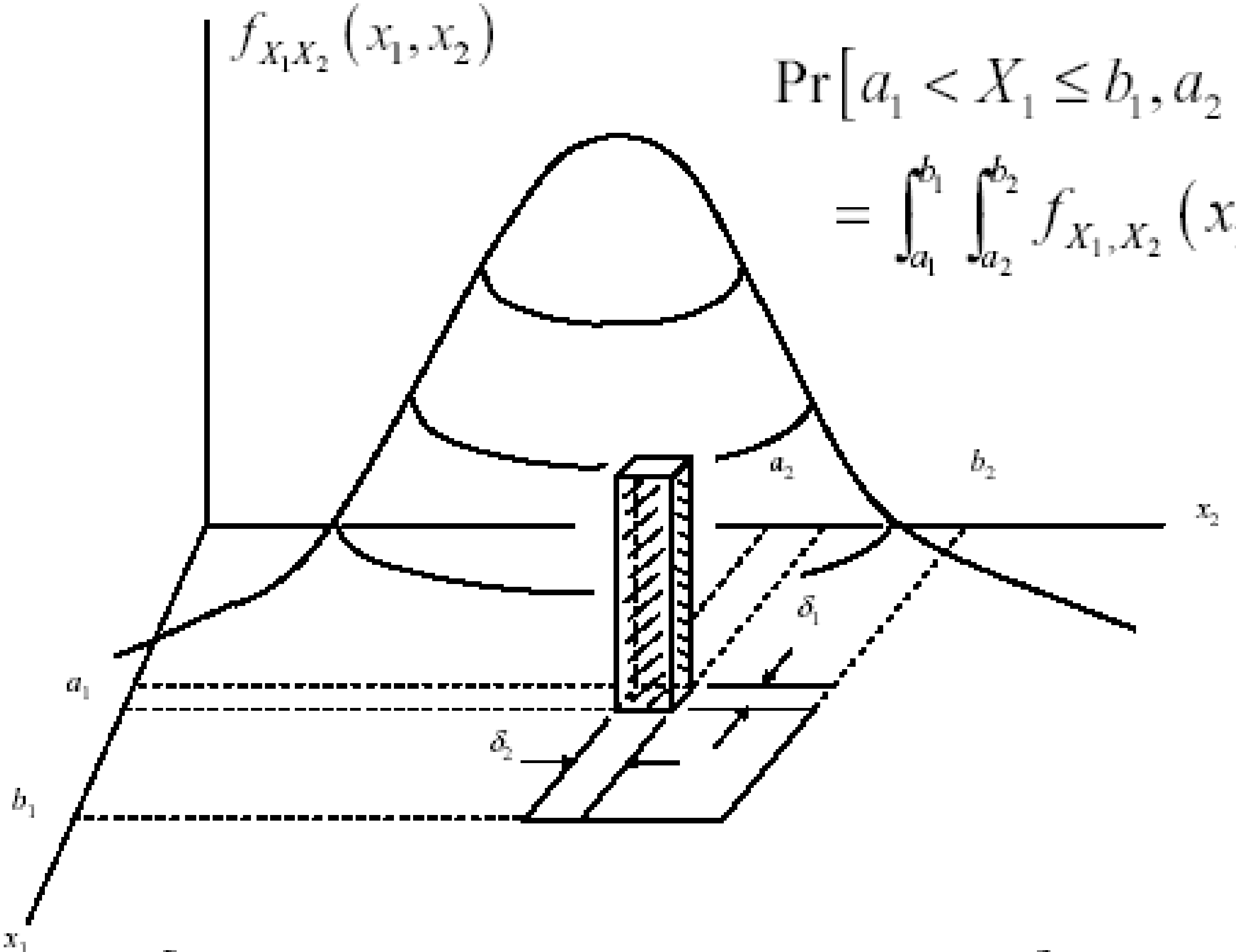
- Two random variables X and Y have a common probability density functions as :

$$f_{XY}(x,y) = ce^{-(x^2+y^2)}$$

- *(X,Y)→ $f_{XY}(x,y)$ is the probability density function of the couple (X,Y)*
- *Example:*



Interpretation of joint PDF as probability:




$$\Pr\left[a_1 < X_1 \le b_1, a_2 < X_2 \le b_2\right] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

$$\Pr\left[a_1 < X_1 \le a_1+\delta_1, a_2 < X_2 \le a_2+\delta_2\right] \cong f_{X_1,X_2}(a_1,a_2)\,\delta_1\delta_2$$

# Bi-dimensional Random variables

- Cumulative functions:

$$F_{X,Y}(x,y) = P(X < x, Y < y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x,y)\,dxdy$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dxdy = 1$$

- Marginal cumulative distribution functions

$$F_X(x) = F_{XY}(x,+\infty) = \int_{-\infty}^{x}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dxdy$$

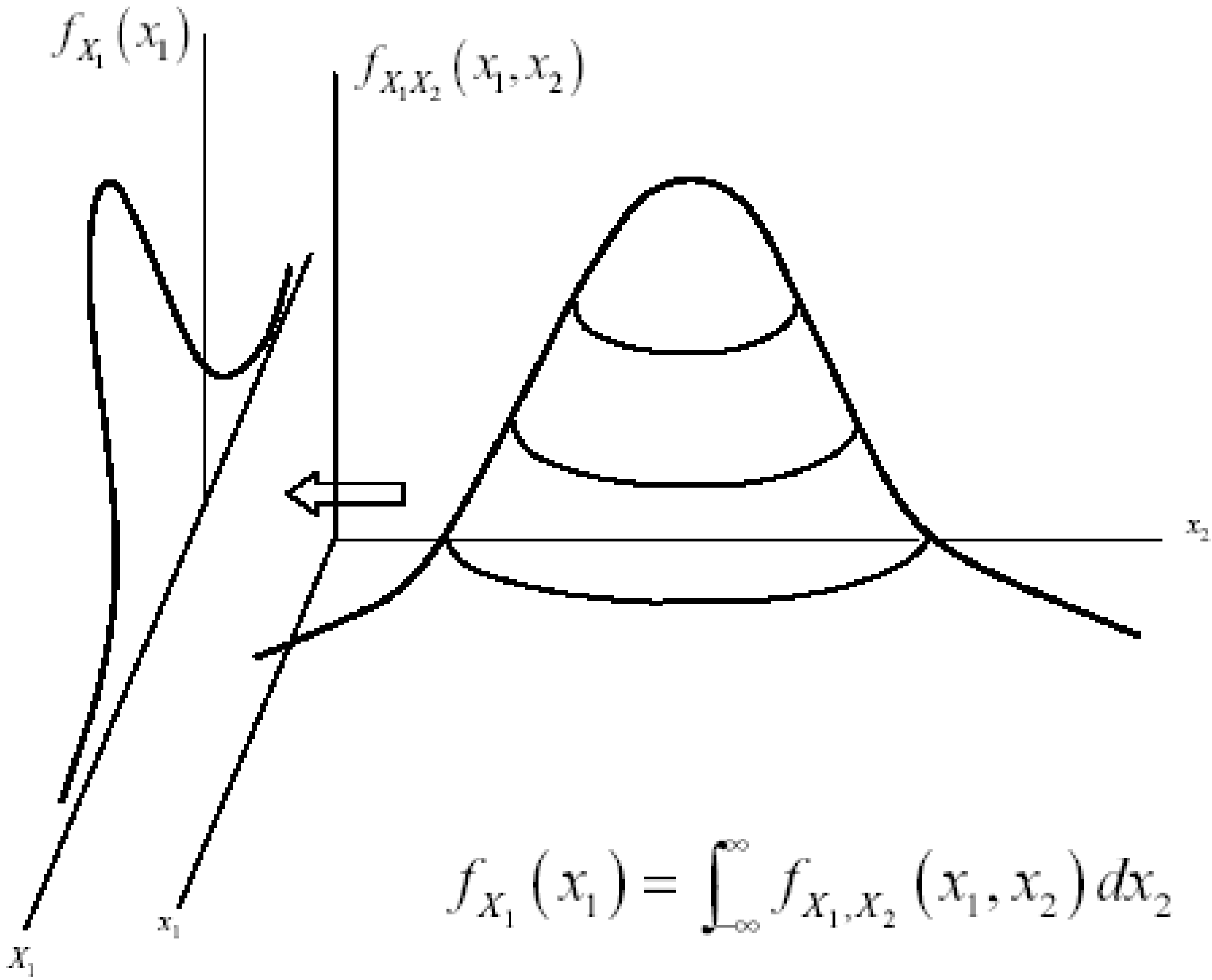
$$F_Y(y) = F_{XY}(+\infty,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{y} f_{X,Y}(x,y)\,dxdy$$

- Marginal probability density functions

$$f_X(u) = \int_{-\infty}^{+\infty} f_{XY}(u,v)\,dv$$

$$f_X(v) = \int_{-\infty}^{+\infty} f_{XY}(u,v)\,du$$

Interpretation of marginal pdf as a projection:




$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_2$$

**Example:**

$$f_{X_1X_2}(x_1,x_2)=\begin{cases} ce^{-x_1}e^{-2x_2}, & 0\le x_1\le x_2<\infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $c$

$$1=c\int_0^{\infty}\int_0^{x_2}e^{-x_1}e^{-2x_2}\,dx_1dx_2=c\int_0^{\infty}\left(1-e^{-x_2}\right)e^{-2x_2}\,dx_2$$

$$1=c\left[\frac{e^{-2x_2}}{-2}-\frac{e^{-3x_2}}{-3}\right]_0^{\infty}$$

$$\frac{c}{6}=1 \quad\Rightarrow\quad c=6$$