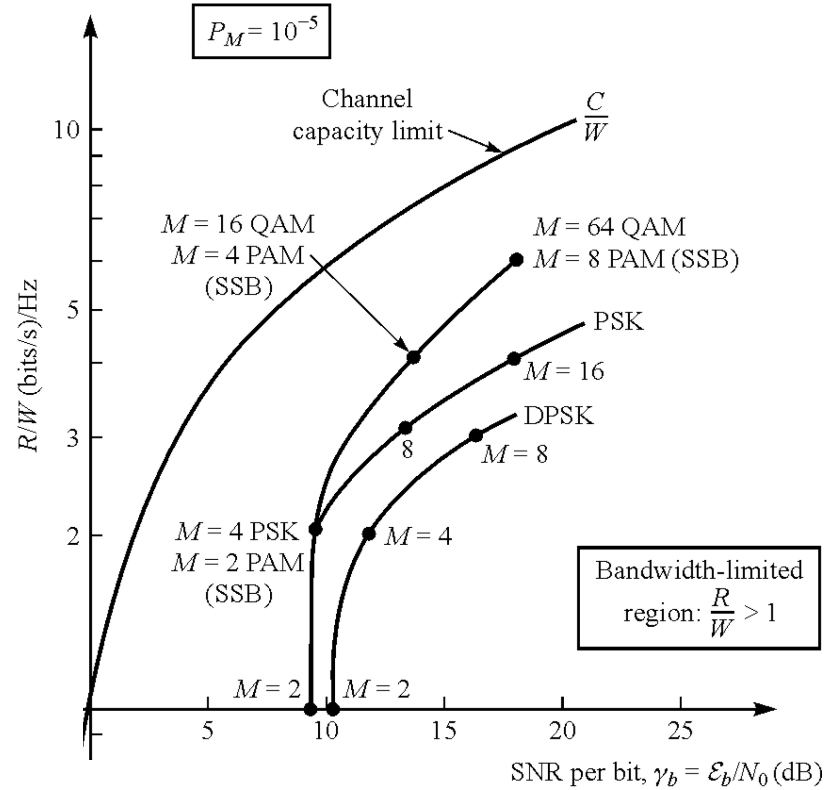
Channel Capacity

- Discrete Memoryless Channels
- Random Codes
- Block Codes
- Trellis Codes

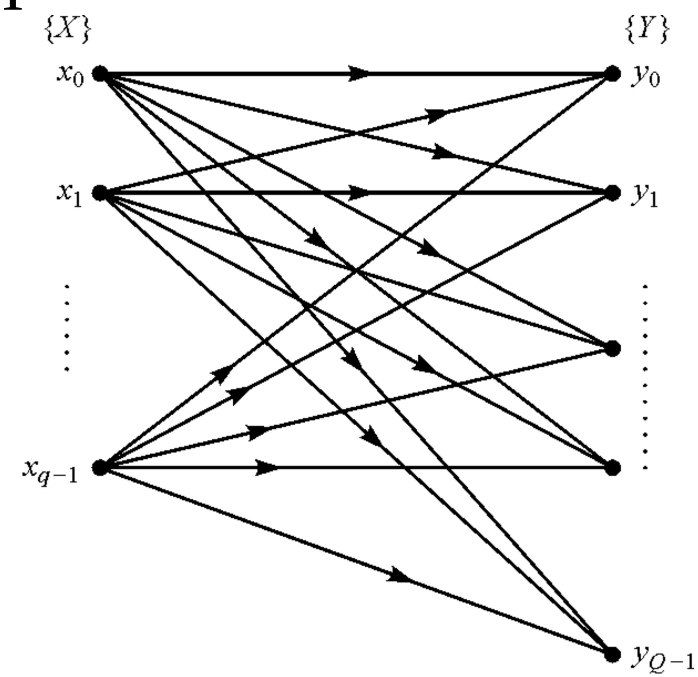


Channel Models

- Discrete Memoryless Channel
  - Discrete-discrete
    - Binary channel, M-ary channel
  - Discrete-continuous
    - M-ary channel with soft-decision (analog)
  - Continuous-continuous
    - Modulated waveform channels (QAM)

Discrete Memoryless Channel

- Discrete-discrete
  - Binary channel, M-ary channel



Probability transition matrix

$$
\mathbf{P} = \begin{bmatrix} P(Y = y_1 \mid X = x_1) & . & . & . \\ . & . & P(y_i \mid x_j) = p_{ji} & . \\ . & . & . & . \\ . & . & . & p_{q-1Q-1} \end{bmatrix}
$$

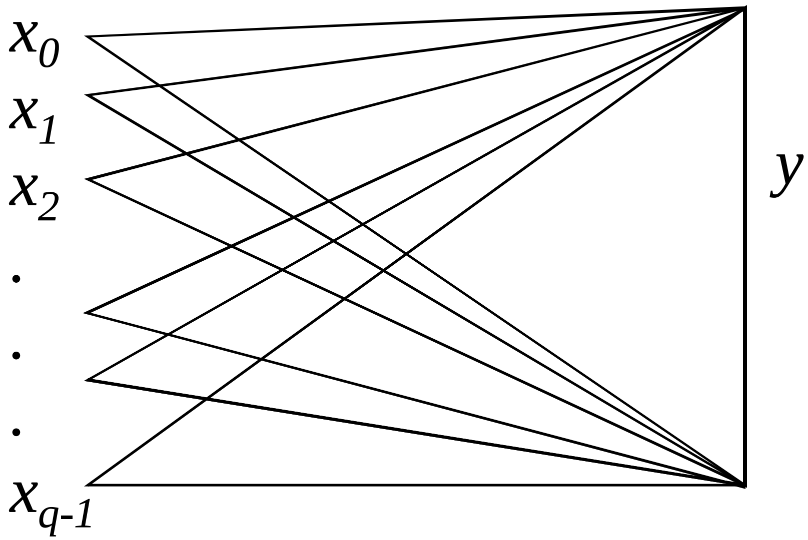
Discrete Memoryless Channel

- Discrete-continuous
  - M-ary channel with soft-decision (analog) output

AWGN

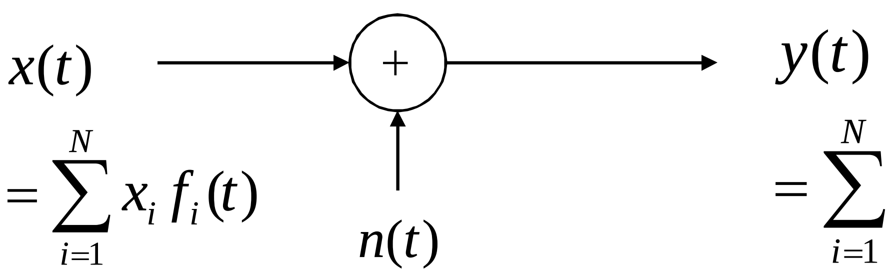
$$p(y\,|\,x_k) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(y-x_k)^2/2\sigma^2}$$

$$\mathbf{P} = \begin{bmatrix} p(y\,|\,X = x_1) \\ p(y\,|\,X = x_k) \end{bmatrix}$$



Discrete Memoryless Channel

- Continuous-continuous
  - Modulated waveform channels (QAM)
  - Assume Band limited waveforms, bandwidth = *W*
    - Sampling at Nyquist = *2W* sample/s
  - Then over interval of *N = 2WT* samples use an orthogonal function expansion:



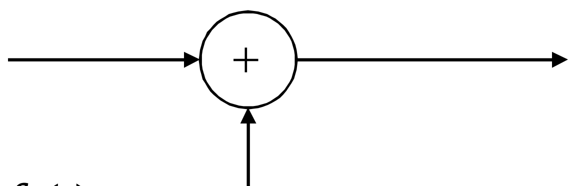
$$x(t) = \sum_{i=1}^{N} x_i f_i(t)$$

$$n(t) = \sum_{i=1}^{N} n_i f_i(t)$$

$$y(t) = \sum_{i=1}^{N} y_i f_i(t)$$

Discrete Memoryless Channel

- Continuous-continuous
  - Using orthogonal function expansion:



$$x(t) = \sum_{i=1}^{N} x_i f_i(t)$$

$$n(t) = \sum_{i=1}^{N} n_i f_i(t)$$

$$\begin{aligned} y(t) &= \sum_{i=1}^{N} y_i f_i(t) \\ &= \sum_{i=1}^{N} \left\{ \int_0^T y(t) f_i^*(t) dt \right\} f_i(t) \\ &= \sum_{i=1}^{N} \left\{ \int_0^T \left[x(t) + n(t) \right] f_i^*(t) dt \right\} f_i(t) \\ &= \sum_{i=1}^{N} \left[x_i + n_i \right] f_i(t) \end{aligned}$$

Discrete Memoryless Channel

- Continuous-continuous
  - Using orthogonal function expansion get an equivalent discrete time channel:

$$y_1 = x_1 + n_1$$

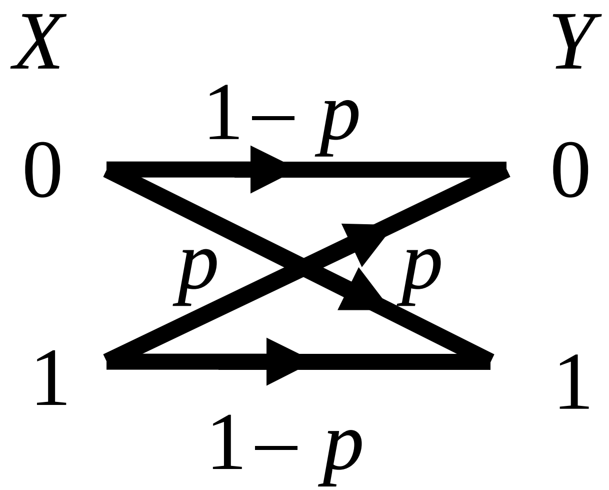
← Gaussian noise

$$p(y_i \mid x_i) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{-(y_i - x_i)^2 / 2\sigma_i^2}$$

x_1	y_1
.	y_2
.	.
.	.
.	.
x_N	y_N

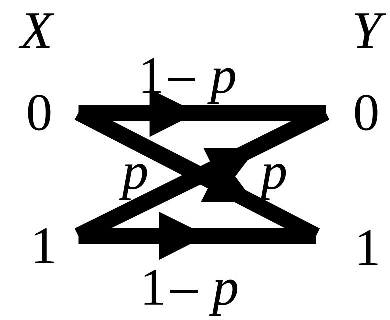
Capacity of binary symmetric channel

- BSC $X = \{0,1\}$ $Y = \{0,1\}$



$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

Capacity of binary symmetric channel



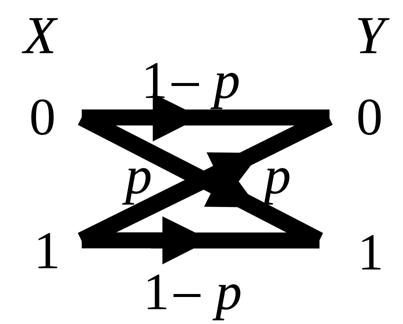
- Average Mutual Information

$$
\begin{aligned}
I(X;Y) = {} & P(X=0)P(Y=0 \mid X=0)\log\frac{P(Y=0 \mid X=0)}{P(Y=0)} + \cdots \\
& P(X=0)P(Y=1 \mid X=0)\log\frac{P(Y=1 \mid X=0)}{P(Y=1)} + \cdots \\
& P(X=1)P(Y=0 \mid X=1)\log\frac{P(Y=0 \mid X=1)}{P(Y=0)} + \cdots \\
& P(X=1)P(Y=1 \mid X=1)\log\frac{P(Y=1 \mid X=1)}{P(Y=1)} \\
= {} & P(X=0)(1-p)\log\frac{1-p}{(1-p)P(X=0)+pP(X=1)} + \cdots \\
& P(X=0)\,p\log\frac{p}{pP(X=0)+(1-p)P(X=1)} + \cdots \\
& P(X=1)\,p\log\frac{p}{(1-p)P(X=0)+pP(X=1)} + \cdots \\
& P(X=1)(1-p)\log\frac{1-p}{pP(X=0)+(1-p)P(X=1)}
\end{aligned}
$$

Capacity of binary symmetric channel

- Channel Capacity is Maximum Information
  - earlier showed: $\max(I(X;Y)) \Leftrightarrow P(X=1) = P(X=0) = \frac{1}{2}$

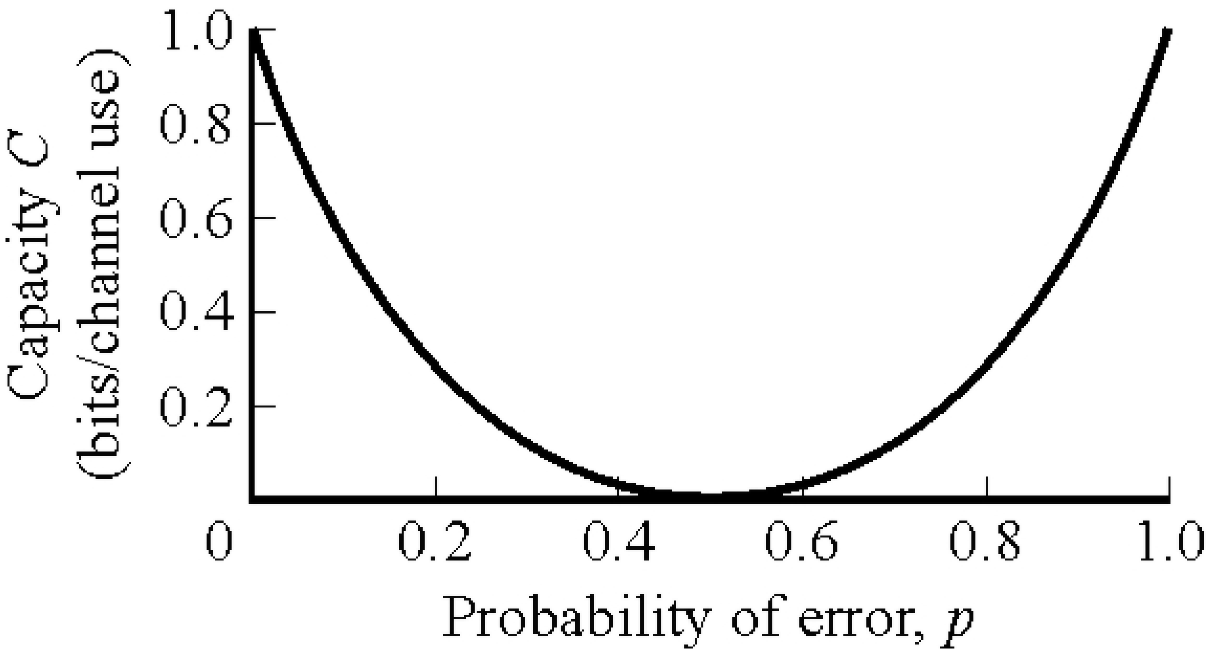
$$
\begin{aligned}
C = \max(I(X;Y)) = \Bigg[& P(X=0)(1-p)\log\frac{1-p}{(1-p)P(X=0)+pP(X=1)} + \cdots \\
& P(X=0)\,p\log\frac{p}{pP(X=0)+(1-p)P(X=1)} + \cdots \\
& P(X=1)\,p\log\frac{p}{(1-p)P(X=0)+pP(X=1)} + \cdots \\
& P(X=1)(1-p)\log\frac{1-p}{pP(X=0)+(1-p)P(X=1)} \Bigg]\Bigg|_{P(X=1)=P(X=0)=\frac{1}{2}}
\end{aligned}
$$

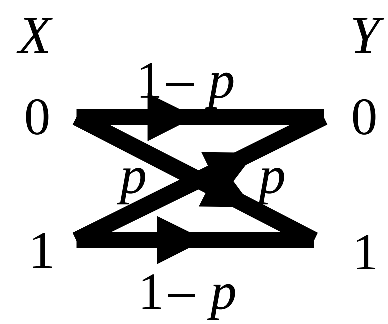


$$= (1-p)\log 2(1-p) + p\log 2p$$

Capacity of binary symmetric channel

- Channel Capacity
  - When *p*=1 bits are inverted but information is perfect if invert them back!

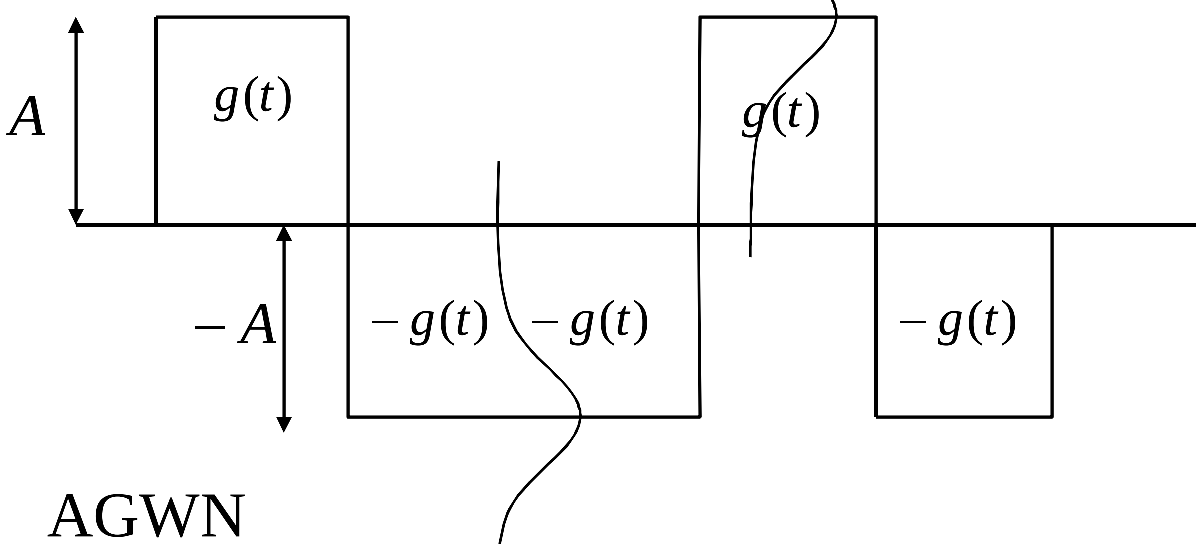




$$C = (1-p)\log_2 2(1-p) + p\log_2 2p$$

Capacity of binary symmetric channel

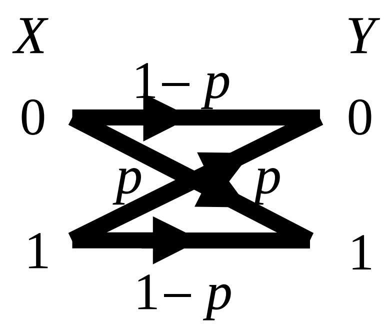
- Effect of SNR on Capacity
  - Binary PAM signal (digital signal amplitude $2A$)



AGWN

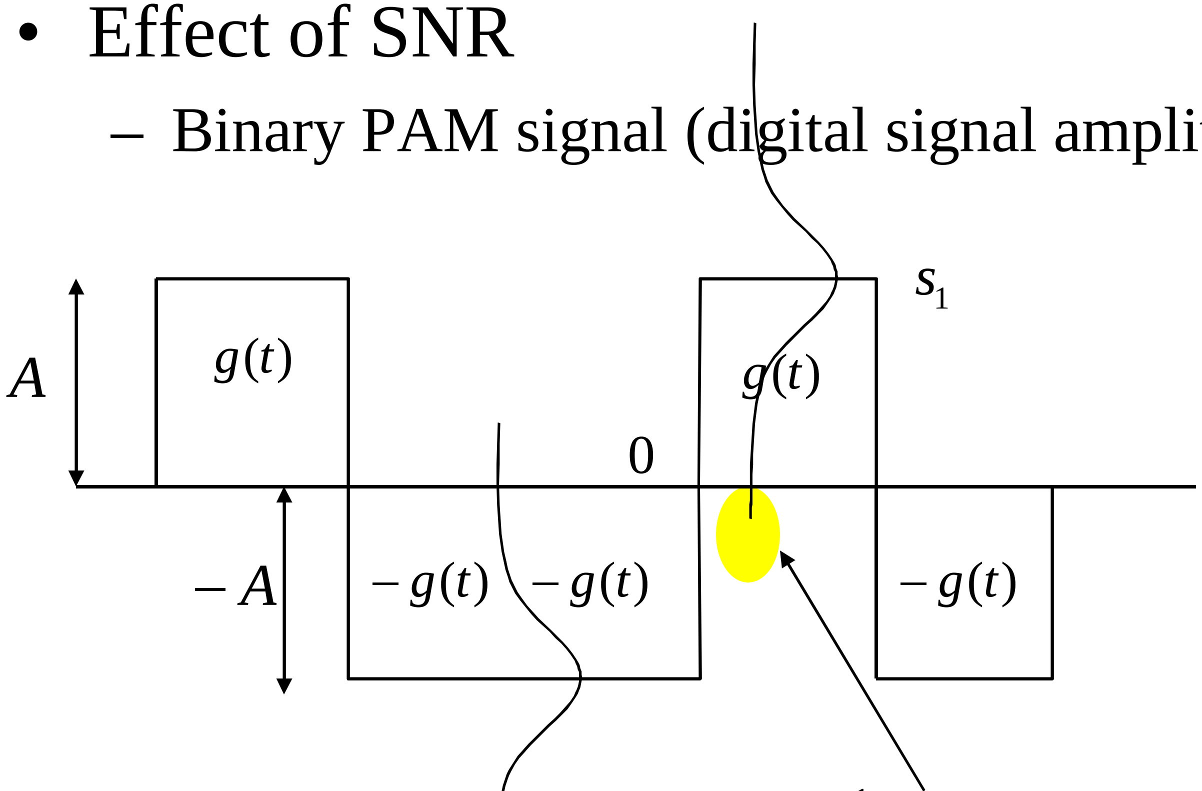
$$p(r \mid s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0}$$

$$p(r \mid s_2) = \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0}$$



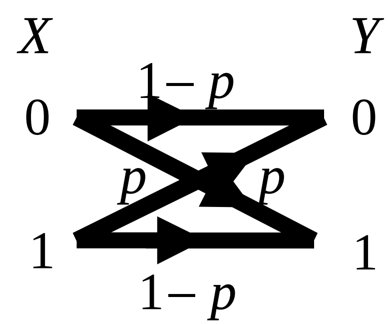
Capacity of binary symmetric channel

- Effect of SNR
  – Binary PAM signal (digital signal amplitude 2*A*)



$$P(e \mid s_1) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr$$

$$= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = P(e \mid s_2)$$



Capacity of binary symmetric channel

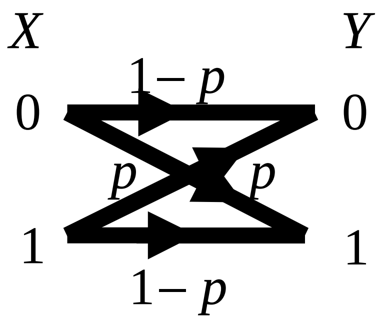
- Effect of SNR
  – Binary PAM signal (digital signal amplitude 2*A*)

$$P_b = \tfrac{1}{2} P(e \mid s_1) + \tfrac{1}{2} P(e \mid s_2)$$

$$= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

$$= Q\left(\sqrt{2SNR_b}\right) = Q\left(\sqrt{2\gamma_b}\right)$$

$$= Q\left(2A\sqrt{\frac{1}{2N_0}}\right)$$



Capacity of binary symmetric channel

- Effect of SNR
  - Binary PAM signal (digital signal amplitude 2*A*)

$$\text{rms noise} = \sigma = \sqrt{\frac{N_0}{2}}$$

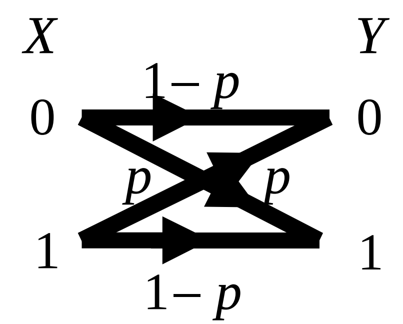
Not sure about this
Does it depend on bandwidth?

$$\Rightarrow$$

$$P_b = Q\left(2A\sqrt{\frac{1}{2N_0}}\right)$$

$$= Q\left(2A\sqrt{\frac{2}{4N_0}}\right)$$

$$= Q\left(\tfrac{1}{2}\frac{2A}{\sigma}\right) = Q\left(\tfrac{1}{2}\frac{2A}{\text{rms noise}}\right)$$

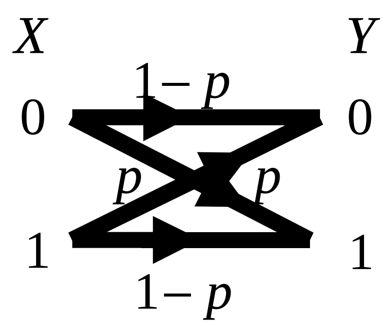


Capacity of binary symmetric channel

- Effect of SNR
  - Binary PAM signal (digital signal amplitude 2A)

$$P_b = p = Q\left(\frac{1}{2}\frac{\text{Amplitude}}{\text{rms noise}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\frac{\text{Amplitude}}{\sqrt{2}\ \text{rms noise}}\right)$$

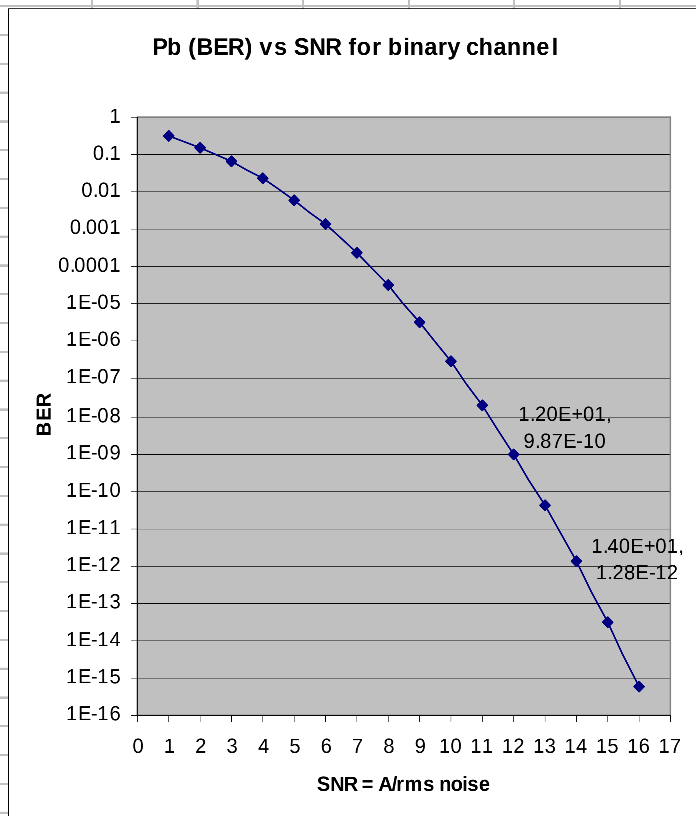


$$\text{C} = (1-p)\log 2(1-p) + p\log 2p$$

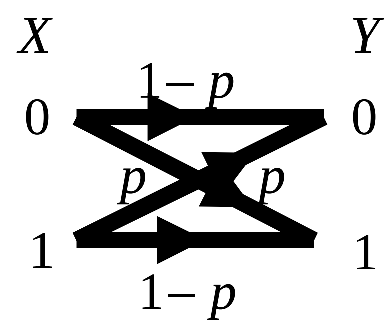
Capacity of binary symmetric channel

- Effect of SNR
  - Binary PAM signal (digital signal amplitude $2A$)

SNR	Pb
1	0.308538
2	0.158655
3	0.066807
4	0.02275
5	0.00621
6	0.00135
7	0.000233
8	3.17E-05
9	3.4E-06
10	2.87E-07
11	1.9E-08
12	9.87E-10
13	4.02E-11
14	1.28E-12
15	3.19E-14
16	6.11E-16



$$P_b = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\frac{\text{Amplitude}}{\sqrt{2}\text{ rms noise}}\right)$$

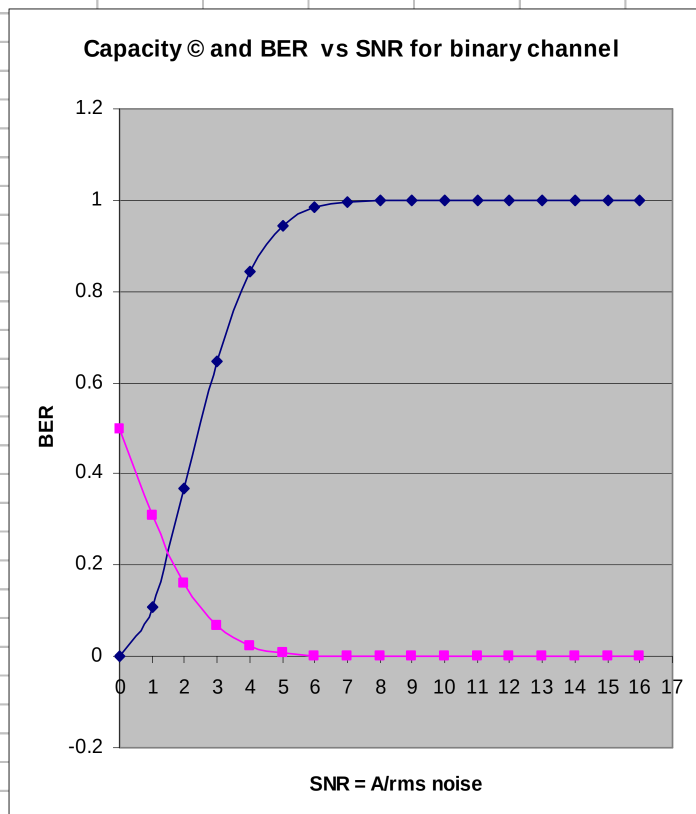


Capacity of binary symmetric channel

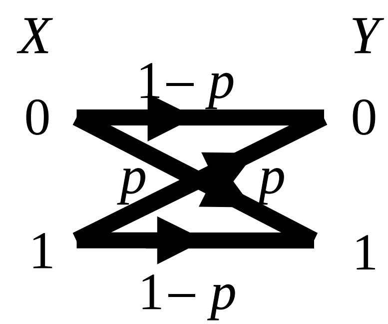
- Effect of SNR
  - Binary PAM signal (digital signal amplitude 2*A*)

$$C = (1-p)\log_2 2(1-p) + p\log_2 2p$$

SNR	Pb	C
0	0.5	0
1	0.308538	0.108522
2	0.158655	0.368917
3	0.066807	0.646106
4	0.02275	0.843385
5	0.00621	0.945544
6	0.00135	0.985185
7	0.000233	0.996857
8	3.17E-05	0.999481
9	3.4E-06	0.999933
10	2.87E-07	0.999993
11	1.9E-08	0.999999
12	9.87E-10	1
13	4.02E-11	1
14	1.28E-12	1
15	3.19E-14	1
16	6.11E-16	1



$$P_b = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\frac{\text{Amplitude}}{\sqrt{2}\ \text{rms noise}}\right)$$



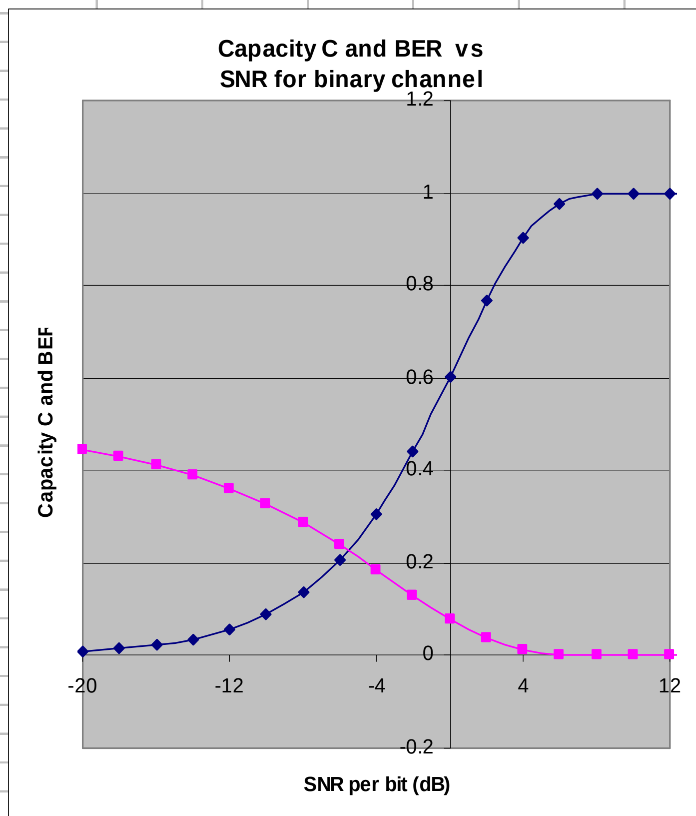
At capacity SNR = 7, so waste lots of SNR to get low BER!!!

Capacity of binary symmetric channel

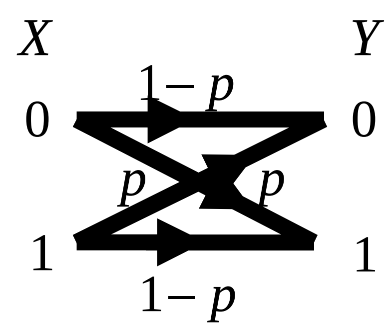
- Effect of SNR_b
  - Binary PAM signal (digital signal amplitude $2A$)

$$C = (1-p)\log_2 2(1-p) + p\log_2 2p$$

SNRb (dB)	Pb	C
-20	0.443769	0.009143
-18	0.429346	0.014452
-16	0.411325	0.022809
-14	0.388906	0.03591
-12	0.361207	0.056319
-10	0.32736	0.087793
-8	0.286715	0.135561
-6	0.239229	0.206245
-4	0.186114	0.306729
-2	0.130645	0.440797
0	0.07865	0.602597
2	0.037506	0.769261
4	0.012501	0.90305
6	0.002388	0.975757
8	0.000191	0.997366
10	3.87E-06	0.999925
12	9.01E-09	1
14	6.81E-13	1

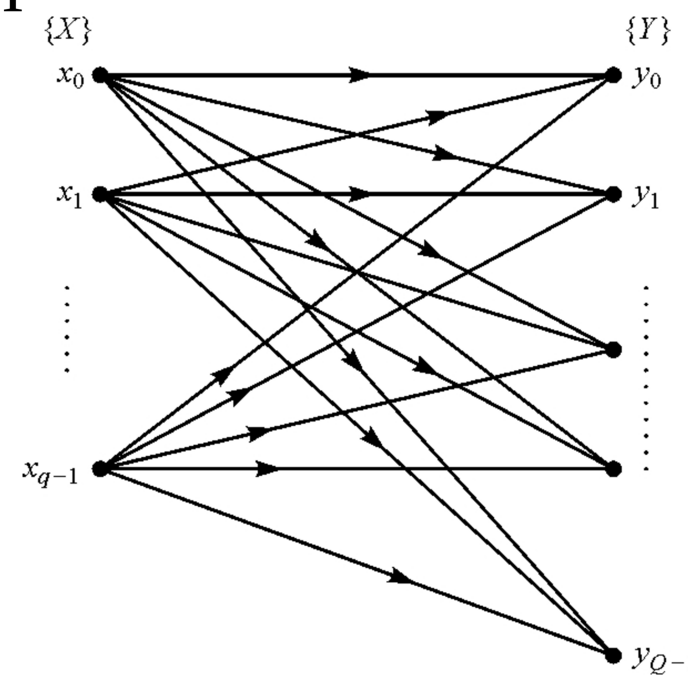


$$P_b = p = Q\left(\sqrt{2\gamma_b}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\gamma_b}\right)$$



Channel Capacity of Discrete Memoryless Channel

- Discrete-discrete
  - Binary channel, M-ary channel

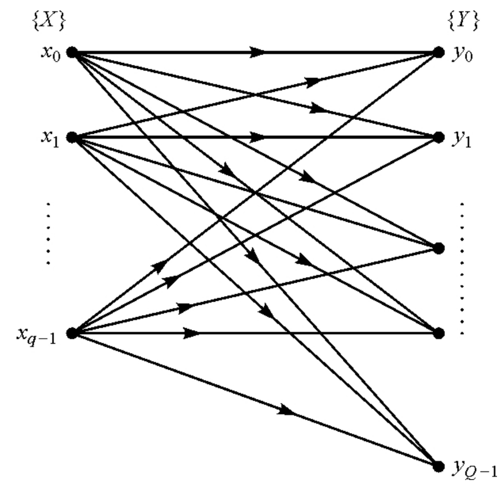


Probability transition matrix

$$\mathbf{P} = \begin{bmatrix} P(Y=y_1 \mid X=x_1) & . & . & . \\ . & . & P(y_i \mid x_j) = p_{ji} & . \\ . & . & . & . \\ . & . & . & p_{q-1Q-1} \end{bmatrix}$$

Channel Capacity of Discrete Memoryless Channel

Average Mutual Information



$$I(X;Y)=\sum_{j=0}^{q-1}\sum_{i=1}^{Q-1}P(X=x_j)P(Y=y_i\mid X=x_j)\log\frac{P(Y=y_i\mid X=x_j)}{P(Y=y_j)}$$

$$\equiv\sum_{j=0}^{q-1}\sum_{i=1}^{Q-1}P(x_j)P(y_i\mid x_j)\log\frac{P(y_i\mid x_j)}{P(y_j)}$$

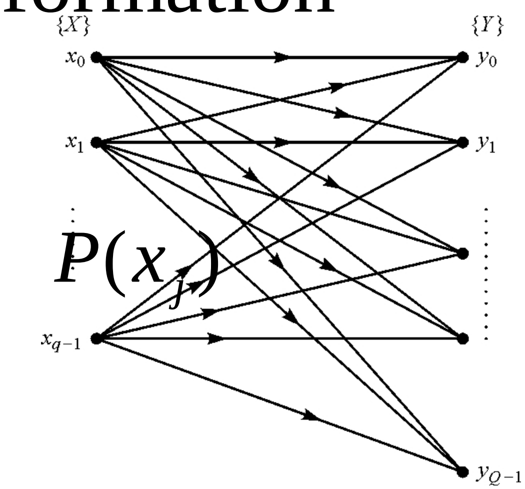
Channel Capacity of Discrete Memoryless Channel

Channel Capacity is Maximum Information

Occurs for $P(x_j) = p,$ for all j

only if $\mathbf{P} =$ symmetric

Otherwise must work out max $P(x_j)$

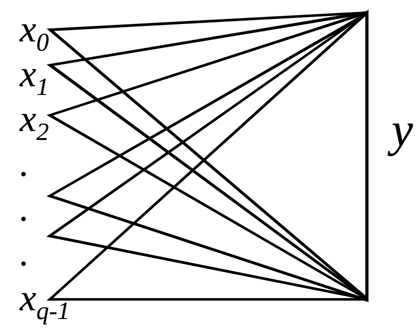


$$C = \max_{P(x_j)}\left(I(X;Y)\right) = \max_{P(x_j)}\left(\sum_{j=0}^{q-1}\sum_{i=1}^{Q-1} P(X=x_j)P(Y=y_i \mid X=x_j)\log\frac{P(Y=y_i \mid X=x_j)}{P(Y=y_j)}\right)$$

$$\equiv \max_{P(x_j)}\left(\sum_{j=0}^{q-1}\sum_{i=1}^{Q-1} P(x_j)P(y_i \mid x_j)\log\frac{P(y_i \mid x_j)}{P(y_j)}\right)\Bigg|_{P(x_j)\geq 0,\quad \sum_{j=0}^{q-1} P(x_j)=1}$$

Channel Capacity Discrete Memoryless Channel

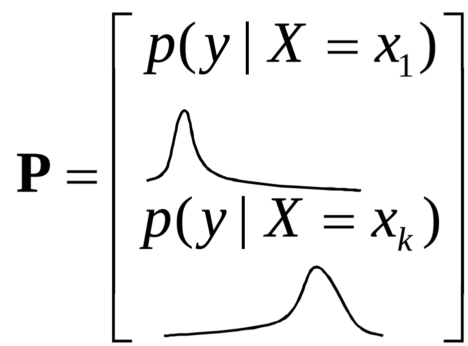
- Discrete-continuous
- Channel Capacity



$$C = \max_{P(x_i)}\big(I(X;Y)\big) = \max_{P(x_i)}\left(\sum_{i=0}^{q-1}\int_{-\infty}^{\infty} P(X=x_i)\,p(Y=y\mid X=x_i)\log\frac{p(Y=y\mid X=x_i)}{p(Y=y)}\,dy\right)$$

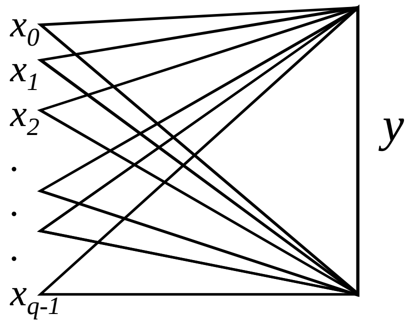
where

$$p(Y=y) = \sum_{i=0}^{q-1} P(X=x_i)\,p(Y=y\mid X=x_i)$$



Channel Capacity Discrete Memoryless Channel

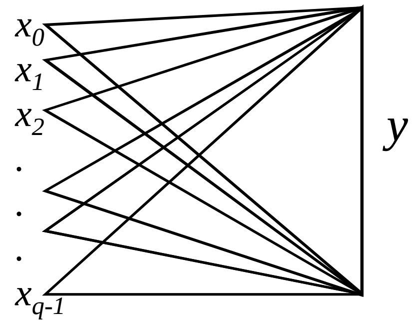
- Discrete-continuous
- Channel Capacity with AWGN



$$p(y \mid x_k) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(y-x_k)^2/2\sigma^2}$$

$$C = \max_{P(x_i)} \left(\sum_{i=0}^{q-1} \int_{-\infty}^{\infty} P(X = x_i) \frac{1}{\sqrt{2\pi\sigma}} e^{-(y-x_i)^2/2\sigma^2} \log \frac{\frac{1}{\sqrt{2\pi\sigma}} e^{-(y-x_i)^2/2\sigma^2}}{\sum_{i=0}^{q-1} P(X = x_i) \frac{1}{\sqrt{2\pi\sigma}} e^{-(y-x_i)^2/2\sigma^2}} dy \right)$$

Channel Capacity Discrete Memoryless Channel



- Binary Symmetric PAM-continuous
- Maximum Information when:

$$P(X = A) = P(X = -A) = \tfrac{1}{2}$$

$$C = \tfrac{1}{2}\frac{1}{\sqrt{2\pi\sigma}}\left[\log\frac{2e^{2A^2/2\sigma^2}}{e^{A^2/2\sigma^2}+e^{-A^2/2\sigma^2}}\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}dy\right.$$

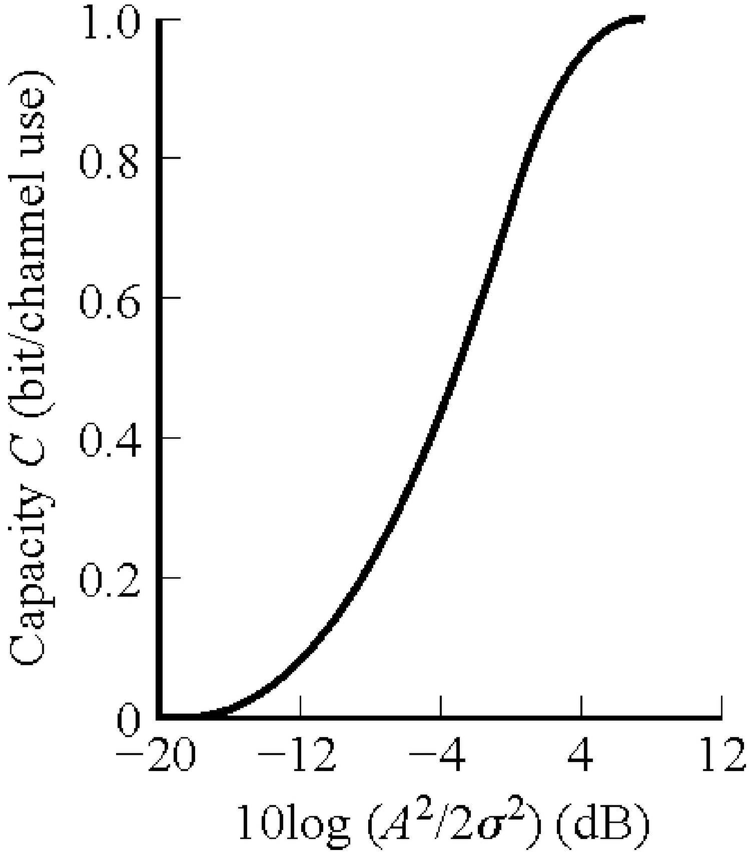
$$\left.+\log\frac{2e^{-2A^2/2\sigma^2}}{e^{A^2/2\sigma^2}+e^{-A^2/2\sigma^2}}\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}dy\right]$$

Channel Capacity Discrete Memoryless Channel

- Binary Symmetric PAM-continuous
- Maximum Information when:

$$C = \tfrac{1}{2}\frac{1}{\sqrt{2\pi\sigma}}\left[\log\frac{2e^{2A^2/2\sigma^2}}{e^{A^2/2\sigma^2}+e^{-A^2/2\sigma^2}}\int_{-\infty}^{\infty}e^{-y^2/2\sigma^2}dy\right.$$

$$\left.+\log\frac{2e^{-2A^2/2\sigma^2}}{e^{A^2/2\sigma^2}+e^{-A^2/2\sigma^2}}\int_{-\infty}^{\infty}e^{-y^2/2\sigma^2}dy\right]$$



Channel Capacity Discrete Memoryless Channel

- Binary Symmetric PAM-continuous
- Versus Binary Symmetric discrete

SNRb (dB)	Pb	C
-20	0.443769	0.009143
-18	0.429346	0.014452
-16	0.411325	0.022809
-14	0.388906	0.03591
-12	0.361207	0.056319
-10	0.32736	0.087793
-8	0.286715	0.135561
-6	0.239229	0.206245
-4	0.186114	0.306729
-2	0.130645	0.440797
0	0.07865	0.602597
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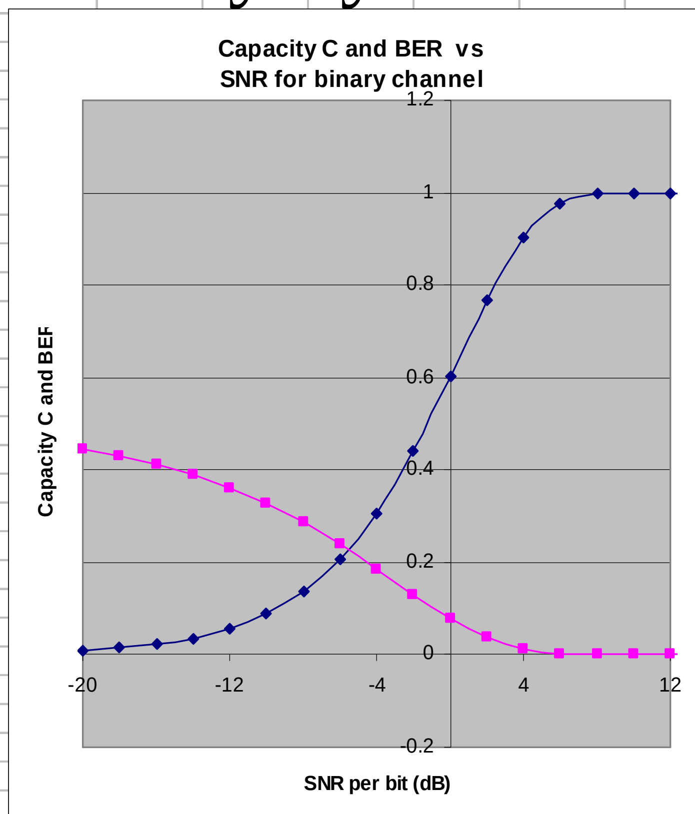
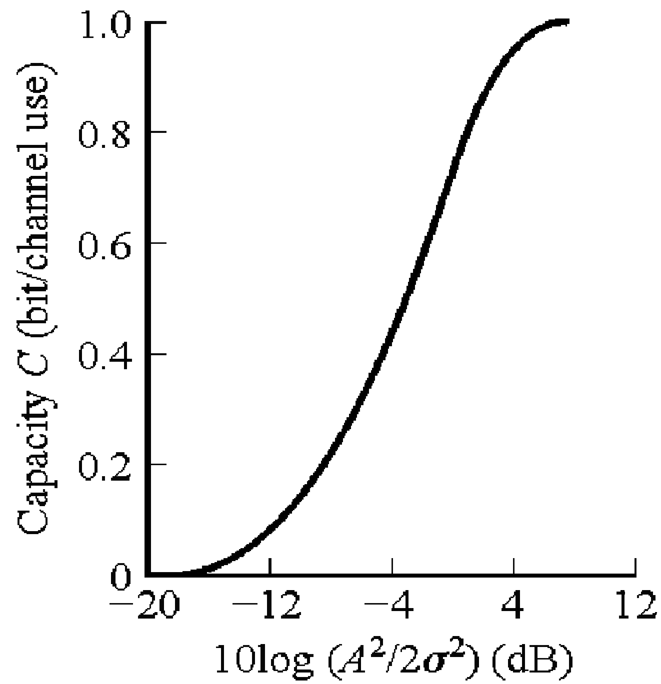
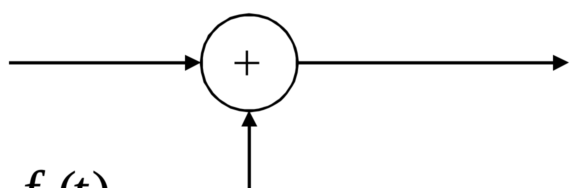
8	0.000191	0.997366
10	3.87E-06	0.999925
12	9.01E-09	1
14	6.81E-13	1





Discrete Memoryless Channel

- Continuous-continuous
  - Modulated waveform channels (QAM)
  - Assume Band limited waveforms, bandwidth = *W*
    - Sampling at Nyquist = *2W* sample/s
  - Then over interval of *N* = *2WT* samples use an orthogonal function expansion:



$$x(t) = \sum_{i=1}^{N} x_i f_i(t)$$

$$y(t) = \sum_{i=1}^{N} y_i f_i(t)$$

$$n(t) = \sum_{i=1}^{N} n_i f_i(t)$$

Discrete Memoryless Channel

- Continuous-continuous
  - Using orthogonal function expansion get an equivalent discrete time channel:

$$y_1 = x_1 + n_1$$

← Gaussian noise

$$p(y_i \mid x_i) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{-(y_i - x_i)^2 / 2\sigma_i^2}$$

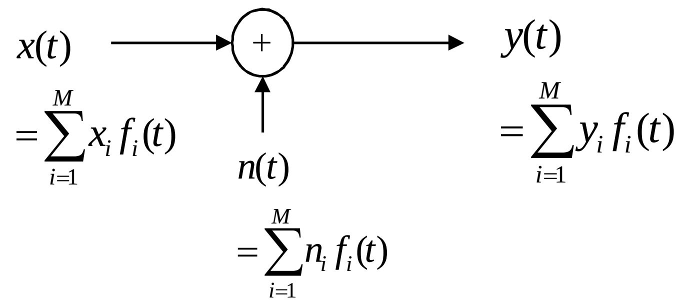
x_1 . . . . x_N

y_1 y_2 . . . y_N

Discrete Memoryless Channel

- Continuous-continuous
- Capacity is (Shannon)

$$p(y_i \mid x_i) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{-(y_i - x_i)^2 / 2\sigma_i^2}$$



$$C = \lim_{T \to \infty} \max_{p(x)} \frac{1}{T} I(X;Y)$$

$$N = 2WT$$

$$I(\mathbf{X}_N;\mathbf{Y}_N) = \int_{\mathbf{X}_N} \cdots \int \int_{\mathbf{Y}_N} \cdots \int p(\mathbf{y}_N \mid \mathbf{x}_N) p(\mathbf{x}_N) \log \frac{p(\mathbf{y}_N \mid \mathbf{x}_N)}{p(\mathbf{y}_N)} d\mathbf{x}_N d\mathbf{y}_N$$

$$= \sum_{i=1}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(y_i \mid x_i) p(x_j) \log \frac{p(y_i \mid x_i)}{p(y_i)} dy_i dx_i$$

Discrete Memoryless Channel

- Continuous-continuous
- Maximum Information when:

$$p(x_i) = \frac{1}{\sqrt{2\pi\sigma_x}} e^{-x_i{}^2/2\sigma_x^2}$$

Statistically independent zero mean Gaussian inputs

then

$$\max_{p(x)} I(\mathbf{X}_N;\mathbf{Y}_N) = \sum_{i=1}^{N} \tfrac{1}{2}\log\left(1+\frac{2\sigma_x^2}{N_0}\right)$$

$$= \frac{1}{2} N \log\left(1+\frac{2\sigma_x^2}{N_0}\right)$$

$$= WT \log\left(1+\frac{2\sigma_x^2}{N_0}\right)$$

Discrete Memoryless Channel

- Continuous-continuous
- Constrain average power in $x(t)$:

$$P_{av} = \frac{1}{T}\int_0^T E[x^2(t)]dt$$

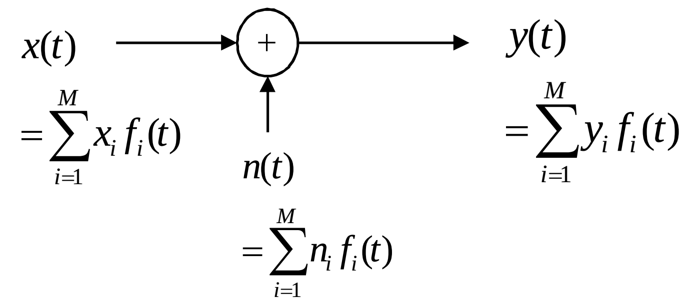
$$= \frac{1}{2}\sum_{i=1}^{N} E(x_i^2)$$

$$= \frac{N\sigma_x^2}{T} = 2W\sigma_x^2$$

Discrete Memoryless Channel

- Continuous-continuous
- Thus Capacity is:

$$p(y_i \mid x_i) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{-(y_i - x_i)^2 / 2\sigma_i^2}$$



$$C = \lim_{T \to \infty} \max_{p(x)} \frac{1}{T} I(\mathbf{X}_N ; \mathbf{Y}_N)$$

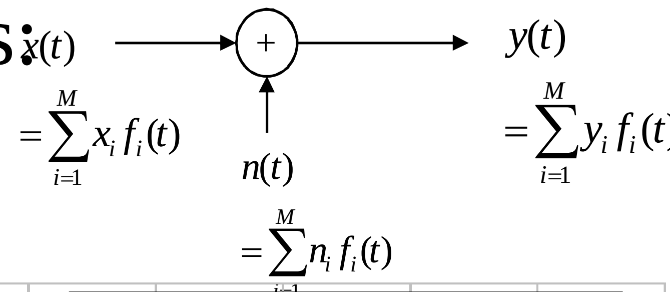
$$= \lim_{T \to \infty} W \log\left(1 + \frac{2\sigma_x^2}{N_0}\right)$$

$$= W \log\left(1 + \frac{P_{av}}{WN_0}\right)$$

Discrete Memoryless Channel

- Continuous-continuous
- Thus Normalized Capacity is:

$$p(y_i \mid x_i) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{-(y_i - x_i)^2 / 2\sigma_i^2}$$



$$\frac{C}{W} = \log_2\left(1 + \frac{P_{av}}{WN_0}\right), \text{ but } P_{av} = C\mathcal{E}_b$$

$$= \log_2\left(1 + \frac{C\mathcal{E}_b}{WN_0}\right)$$

$$\Rightarrow$$

$$\frac{\mathcal{E}_b}{N_0} = \frac{2^{C/W} - 1}{C/W}$$

etab/No (d	C/W
-1.44036	0.1
-1.36402	0.15
-1.24869	0.225
-1.07386	0.3375
-0.8075	0.50625
-0.39875	0.759375
0.234937	1.139063
1.230848	1.708594
2.822545	2.562891
5.41099	3.844336
9.669259	5.766504
16.65749	8.649756
27.92605	12.97463
45.69444	19.46195
73.22669	29.19293
115.4055	43.78939
179.5542	65.68408