Basic Properties of the Fourier Transform

- **Linearity Property:** Given signals $x_1(t)$ and $x_2(t)$ with the Fourier transforms

$$\mathcal{F}[x_1(t)] = X_1(f)$$

$$\mathcal{F}[x_2(t)] = X_2(f).$$

The Fourier transform of $\alpha x_1(t) + \beta x_2(t)$ is

$$\mathcal{F}[\alpha x_1(t) + \beta x_2(t)] = \alpha X_1(f) + \beta X_2(f).$$

- **Duality Property:**

If $X(f) = \mathcal{F}[x(t)]$, then $x(f) = \mathcal{F}[X(-t)]$ and $x(-f) = \mathcal{F}[X(t)]$.

Proof:

$$\begin{aligned}\mathcal{F}[X(-t)] &= \int_{-\infty}^{\infty} X(-t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} X(t)e^{j2\pi ft}dt \\ &= x(f).\end{aligned}$$

$$\begin{aligned}\mathcal{F}[X(t)] &= \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} X(t)e^{j2\pi(-f)t}dt \\ &= x(-f).\end{aligned}$$

- **Time Shift Property:** A shift of t_0 in the time origin causes a phase shift of $-2\pi f t_0$ in the frequency domain.

$$\mathcal{F}[x(t-t_0)] = e^{-j2\pi f t_0}\,\mathcal{F}[x(t)].$$

Proof:

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi f t}dt.$$

Let $t' = t - t_0$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t')e^{-j2\pi f (t'+t_0)}dt'$$

$$= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} x(t')e^{-j2\pi f t'}dt'$$

$$= e^{-j2\pi f t_0}\,\mathcal{F}[x(t)] = e^{-j2\pi f t_0}X(f).$$

- **Scaling Property:** For any real $a \neq 0$, we have

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

- Proof:

Case 1: $a > 0$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

Let $t' = at$; we have

$$dt = (1/a)dt'$$

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(t') e^{-j2\pi(f/a)t'} dt' = \frac{1}{a} X\left(\frac{f}{a}\right).$$

Case 2: $a < 0$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

Let $t' = at$; we have

$$dt = (1/a)dt'$$

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{\infty}^{-\infty} x(t') e^{-j2\pi(f/a)t'} dt' = -\frac{1}{a} X\left(\frac{f}{a}\right).$$

- **Convolution Property:** If the signal $x(t)$ and $y(t)$ both possess Fourier transforms, then

Proof: $\mathcal{F}[x(t)*y(t)]=\mathcal{F}[x(t)]\,\mathcal{F}[y(t)]=X(f)Y(f).$

Convolution
$$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau$$

$$\mathcal{F}[x(t)*y(t)]=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau\right)e^{-j2\pi ft}\,dt$$
$$=\int_{-\infty}^{\infty}x(\tau)\left(\int_{-\infty}^{\infty}y(t-\tau)e^{-j2\pi ft}dt\right)d\tau$$
$$=\int_{-\infty}^{\infty}x(\tau)\left(e^{-j2\pi f\tau}Y(f)\right)d\tau$$
$$=Y(f)\int_{-\infty}^{\infty}x(\tau)e^{-j2\pi f\tau}\,d\tau$$
$$=X(f)Y(f).$$

- **Modulation Property:** The Fourier transform of $x(t)\, e^{j2\pi f_0 t}$ is $X(f - f_0)$, and the Fourier transform of

is $x(t)\cos(2\pi f_0 t) = x(t)\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$

$$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0).$$

Proof:

$$\mathrm{F}\,[x(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f - f_0)t}\,dt$$

$$= X(f - f_0).$$

- **Parseval's Property:** If the Fourier transforms of $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively, then

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df.$$

- proof:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(u)e^{j2\pi ut}du\right)\left(\int_{-\infty}^{\infty} Y(v)e^{j2\pi vt}dv\right)^* dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\,e^{j2\pi ut}\ e^{-j2\pi vt}dt\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\left(\int_{-\infty}^{\infty} e^{j2\pi(u-v)t}dt\right)dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\,\delta(u-v)\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(u)\,\delta(u-v)\,dv\,du$$

$$= \int_{-\infty}^{\infty} X(u)Y^*(u)du$$

$$= \int_{-\infty}^{\infty} X(f)Y^*(f)df.$$

- **Rayleigh's Property:** If *X(f)* is the Fourier transform of *x(t)*, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df .$$

Proof:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} X(f) X^*(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2 \, df .$$

Parseval's Property

- **Autocorrelation Property:** The (time) autocorrelation function of the aperiodic signal *x(t)* is denoted by $R_x(\tau)$ and is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt.$$

The autocorrelation property states that

$$\mathcal{F}\,[R_x(\tau)] = \left|X(f)\right|^2.$$

- **Differentiation Property:** The Fourier transform of the derivative of a signal can be obtained from the relation

$$\mathcal{F}\left[\frac{d}{dt}x(t)\right] = j2\pi f\; X(f).$$

- **Integration Property:** The Fourier transform of the integral of a signal can be determined from the relation

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f).$$

- **Moments Property:** If $\mathcal{F}[x(t)] = X(f)$, then $\int_{-\infty}^{\infty} t^n x(t)dt$, the *n*th moment of *x(t),* can be obtained from the relation

$$\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)\bigg|_{f=0}.$$

TABLE 2.1 TABLE OF FOURIER TRANSFORMS

Time Domain ($x(t)$)	Frequency Domain ($X(f)$)
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f+f_0)+\frac{1}{2j}\delta(f-f_0)$
$\Pi(t)=\begin{cases}1, & \|t\|<\frac{1}{2}\\ \frac{1}{2}, & t=\pm\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$	$\text{sinc}(f)$
$\text{sinc}(t)$	$\Pi(f)$
$\Lambda(t)=\begin{cases}t+1, & -1\le t<0\\ -t+1, & 0\le t<1\\ 0, & \text{otherwise}\end{cases}$	$\text{sinc}^2(f)$
$\text{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)=\begin{cases}1, & t>0\\ -1, & t<0\\ 0, & t=0\end{cases}$	$1/(j\pi f)$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi\,\text{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{T_0}\right)$