MATHEMATICAL MODELS OF SYSTEMS

Introduction

Why?

1) Easy to discuss the full possible types of the control systems —only in terms of the system's "mathematical characteristics".

2) The basis of analyzing or designing the control systems.

What is ?

Mathematical models of systems — the mathematical relation-ships between the system's variables.

How get?

1) theoretical approaches

2) experimental approaches

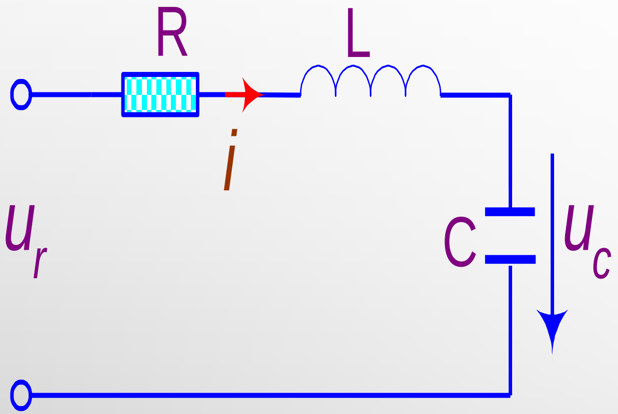
3) discrimination learning

Types

1) Differential equations
2) Transfer function
3) Block diagram、signal flow graph
4) State variables

MATHEMATICAL MODELS OF SYSTEMS

EXAMPLE : A PASSIVE CIRCUIT



$$define: \; input \rightarrow u_r \quad output \rightarrow u_c 。$$

we have:

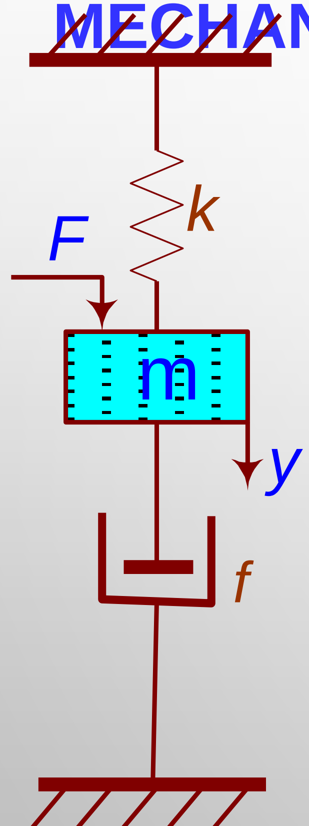
$$Ri + L\frac{di}{dt} + u_c = u_r \quad i = C\frac{du_c}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2u_c}{dt^2} + RC\frac{du_c}{dt} + u_c = u_r$$

$$make: \quad RC = T1 \quad \frac{L}{R} = T2 \quad \Rightarrow \quad T_1T_2\frac{d^2u_c}{dt^2} + T_1\frac{du_c}{dt} + u_c = u_r$$

EXAMPLE : A MECHANISM



Define: *input* $\rightarrow F$, *output* $\rightarrow y$. *We have*:

$$F - ky - f\frac{dy}{dt} = m\frac{d^2y}{d^2t}$$

$$\Downarrow$$

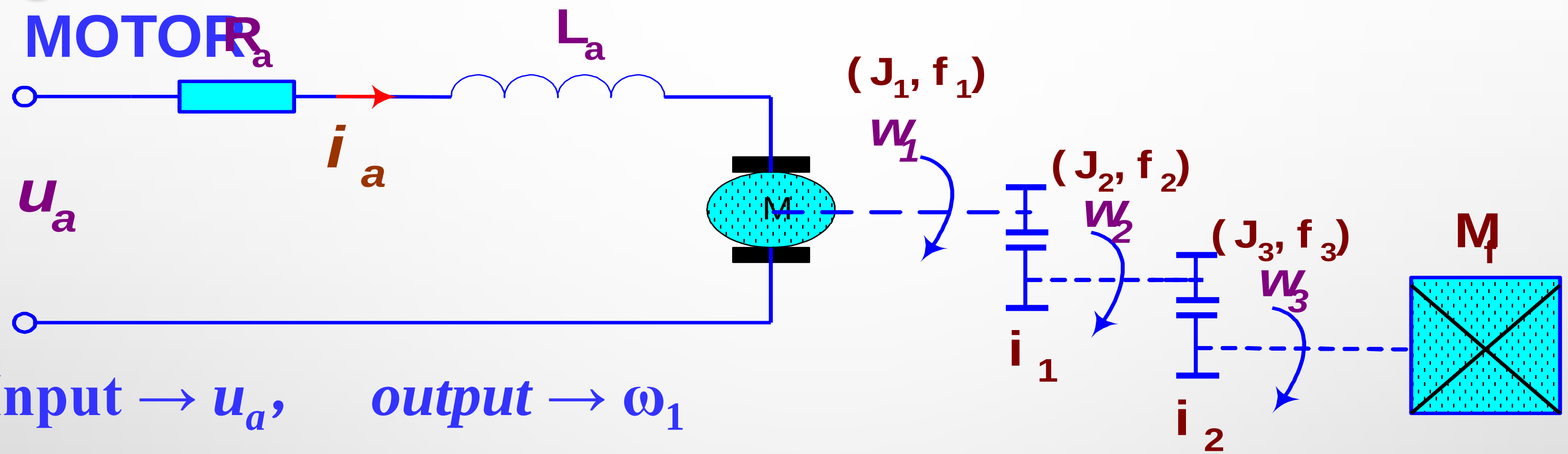
$$m\frac{d^2y}{dt^2} + f\frac{dy}{dt} + ky = F$$

If we make : $\frac{f}{k} = T_1, \quad \frac{m}{f} = T_2$

we have : $T_1T_2\frac{d^2y}{dt^2} + T_1\frac{dy}{dt} + y = \frac{1}{k}F$

Compare with example : $u_c \rightarrow y$, $u_r \rightarrow F$*---analogous systems*

EXAMPLE : A DC MOTOR



Input $\rightarrow u_a$, output $\rightarrow \omega_1$

$$L_a \frac{di_a}{dt} + R_a i_a + E_a = u_a \text{....(1)}$$

$$M = C_m i_a \text{.........................(2)}$$

$$E_a = C_e \omega_1 \text{.........................(3)}$$

$$M - \overline{M} = \overline{J} \frac{d\omega_1}{dt} + \overline{f}\omega_1 \text{.....(4)}$$

(4) → (2) → (1) and (3) → (1) :

$$\frac{L_a \overline{J}}{C_e C_m} \ddot{\omega}_1 + \left(\frac{L_a \overline{f}}{C_e C_m} + \frac{R_a \overline{J}}{C_e C_m}\right)\dot{\omega}_1 + \left(\frac{R_a \overline{f}}{C_e C_m} + 1\right)\omega_1$$

$$= \frac{1}{C_e} u_a - \frac{L_a}{C_e C_m} \dot{\overline{M}} - \frac{R_a}{C_e C_m} \overline{M}$$

$$\overline{J} = J_1 + \frac{J_2}{i_1^2} + \frac{J_3}{i_1^2 i_2^2} \text{......equivalent moment of inertia}$$

$$\text{here: } \overline{f} = f_1 + \frac{f_2}{i_1^2} + \frac{f_3}{i_1^2 i_2^2} \text{......equivalent friction coefficient}$$

$$\overline{M} = \frac{M_f}{i_1 i_2} \text{.........................equivalent torque}$$

(can be derived from: $\omega_1 = i_1\omega_2 = i_1 i_2 \omega_3$)

Make: $T_e = \frac{L_a}{R_a}$ ....electric - magnetic time - constant

$$T_m = \frac{R_a \overline{J}}{C_e C_m} \text{.......mechanical - electric time - constant}$$

$$T_f = \frac{R_a \overline{f}}{C_e C_m} \text{....... friction - electric time - constant}$$

the differential equation description of the DC motor is:

$$T_e T_m \ddot{\omega}_1 + (T_e T_f + T_m)\dot{\omega}_1 + (T_f + 1)\omega_1$$

$$= \frac{1}{C_e} u_a - \frac{1}{J}(T_e T_m \dot{\overline{M}} + T_m \overline{M})$$

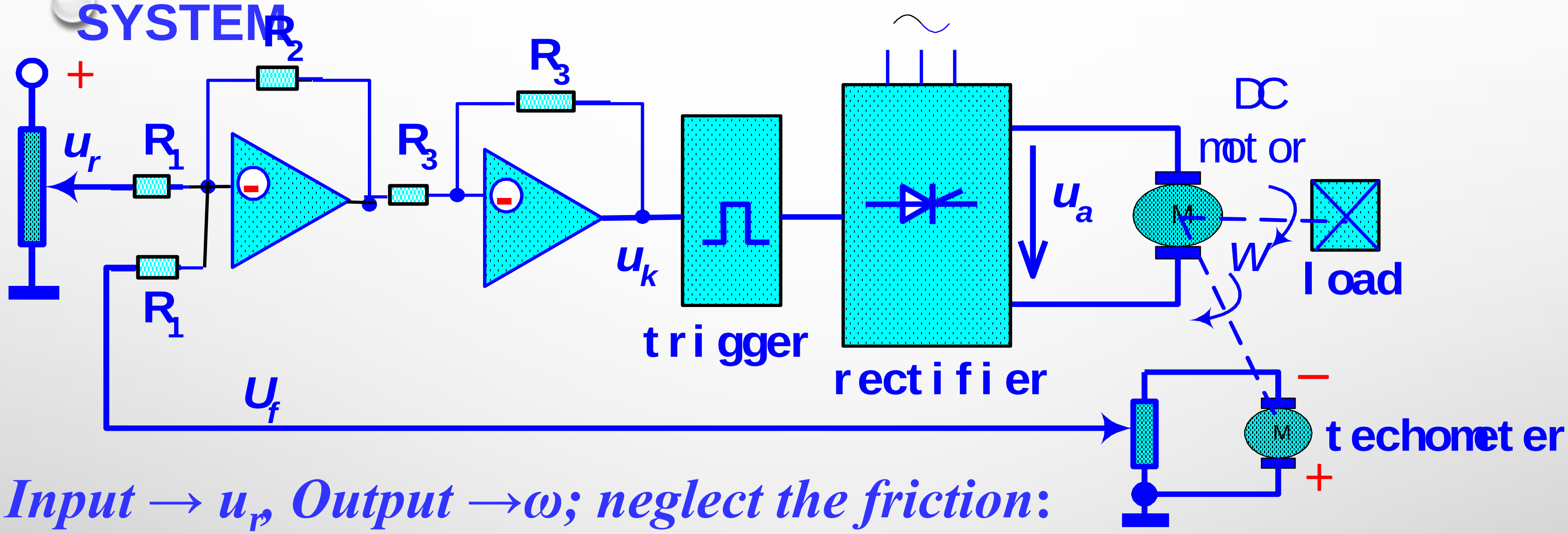
Assume the motor idle: $M_f = 0$, and neglect the friction: $f = 0$, we have:

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \omega = \frac{1}{C_e} u_a$$

Compare with example 2.1 and example 2.2:

$u_c \Leftrightarrow y \Leftrightarrow \omega$; $u_r \Leftrightarrow F \Leftrightarrow u_a$ *----Analogous systems*

EXAMPLE 2.5 : A DC-MOTOR CONTROL SYSTEM



Input → u_r, *Output* → ω; *neglect the friction*:

$$u_k = \frac{R_2}{R_1}(u_r - u_f) = k_1(u_r - u_f)\text{.....................................}(1)$$

$$u_f = \alpha\omega\text{....................}(2) \qquad u_a = k_2 u_k\text{.......................}(3)$$

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \omega = \frac{1}{C_e}u_a - \frac{1}{J}(T_e T_m \dot{\overline{M}} + T_m \overline{M})\text{......}(4)$$

（2）→（1）→（3）→（4），WE HAVE

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \left(1 + k_1 k_2 \alpha \frac{1}{C_e}\right)\omega = k_1 k_2 \frac{1}{C_e} u_r - \frac{T_m}{J}\left(T_e \dot{\overline{M}} + \overline{M}\right)$$

Steps to obtain the input-output description (differential equation) of control systems

1) Identify the output and input variables of the control systems.

2) Write the differential equations of each system's component in terms of the physical laws of the components.
   * necessary assumption and neglect.
   * proper approximation.

3) dispel the intermediate(across) variables to get the input-output description which only contains the output and input variables.

4) Formalize the input-output equation to be the “standard” form:

Input variable —— on the right of the input-output equation .

Output variable —— on the left of the input-output equation.

Writing the polynomial—according to the falling-power order.

General form of the input-output equation of the linear control systems

——A nth-order differential equation:

Suppose: $input \rightarrow r$ ， $output \rightarrow y$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y^{(1)} + a_n y$$
$$= b_0 r^{(m)} + b_1 r^{(m-1)} + b_2 r^{(m-2)} + \cdots + b_{m-1} r^{(1)} + b_m r \text{.........} n \geq m$$

Transfer function

Another form of the input-output(external) description of control systems, different from the differential equations.

Transfer function: ***The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable*** **with all initial condition assumed to be zero and for the linear systems, that is:**

$$G(s) = \frac{C(s)}{R(s)}$$

C(s) —— Laplace transform of the output variable
R(s) —— Laplace transform of the input variable
G(s) —— transfer function

Notes:

- Only for the linear and stationary(constant parameter) systems.
- Zero initial conditions.
- Dependent on the configuration and coefficients of the systems, independent on the input and output variables.

How to obtain the transfer function of a system

1) If the impulse response *g(t)* is known

We have: $$G(s) = L[g(t)]$$

Because: $$G(s) = \frac{C(s)}{R(s)}, \quad \text{if } \; r(t) = \delta(t) \Rightarrow R(s) = 1$$

Then: $$G(s) = C(s) = L[g(t)]$$

Example 2.8 : $$g(t) = 5 - 3e^{-2t} \;\Rightarrow\; G(s) = \frac{5}{s} - \frac{3}{s+2} = \frac{2(s+5)}{s(s+2)}$$

2) If the output response *c(t)* and the input *r(t)* are known

We have: $$G(s) = \frac{L[c(t)]}{L[r(t)]}$$

Example : $r(t) = 1(t) \Rightarrow \mathrm{R(s)} = \frac{1}{s}$ ....*Unit step function*

$$c(t) = 1 - e^{-3t} \Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+3} = \frac{3}{s(s+3)}$$

.........*Unit step response*

Then:

$$G(s) = \frac{C(s)}{R(s)} = \frac{3/s(s+3)}{1/s} = \frac{3}{s+3}$$

3) If the input-output differential equation is known

- **Assume: zero initial conditions;**
- **Make: Laplace transform of the differential equation;**
- **Deduce: *G(s)=C(s)/R(s)*.**

EXAMPLE :

$$2\ddot{c}(t)+3\dot{c}(t)+4c(t)=5\dot{r}(t)+6r(t)$$

$$\Downarrow$$

$$2s^2C(s)+3sC(s)+4C(s)=5sR(s)+6R(s)$$

$$\Downarrow$$

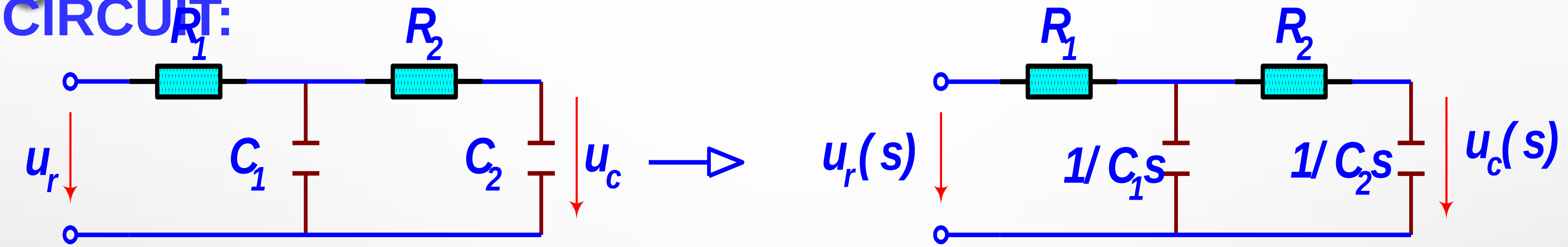
$$G(s)=\frac{C(s)}{R(s)}=\frac{5s+6}{2s^2+3s+4}$$

4) For a circuit

* Transform a circuit into a operator circuit.

* Deduce the *C(s)/R(s)* in terms of the circuits theory.

EXAMPLE : FOR A ELECTRIC CIRCUIT:



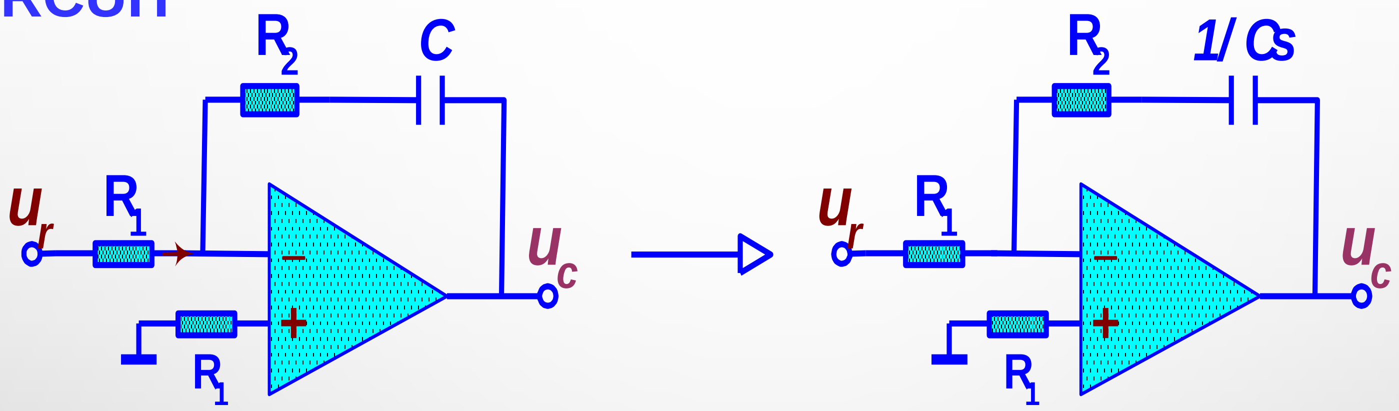
$$U_c(s)=\frac{\frac{1}{sC_1}//(R_2+\frac{1}{sC_2})}{R_1+\frac{1}{sC_1}//(R_2+\frac{1}{sC_2})}\cdot U_r(s)\cdot\frac{\frac{1}{sC_2}}{R_2+\frac{1}{sC_2}}$$

$$=\frac{1}{T_1T_2s^2+(T_1+T_2+T_{12})s+1}\cdot U_r(s)$$

$$G(s)=\frac{U_c(s)}{U_r(s)}=\frac{1}{T_1T_2s^2+(T_1+T_2+T_{12})s+1}$$

here : $T_1=R_1C_1$; $T_2=R_2C_2$; $T_{12}=R_1C_2$

EXAMPLE 2.12: FOR A OP-AMP CIRCUIT



$$G(s)=\frac{U_c(s)}{U_r(s)}=-\frac{R_2+\frac{1}{sC}}{R_1}=-\frac{R_2Cs+1}{R_1Cs}$$

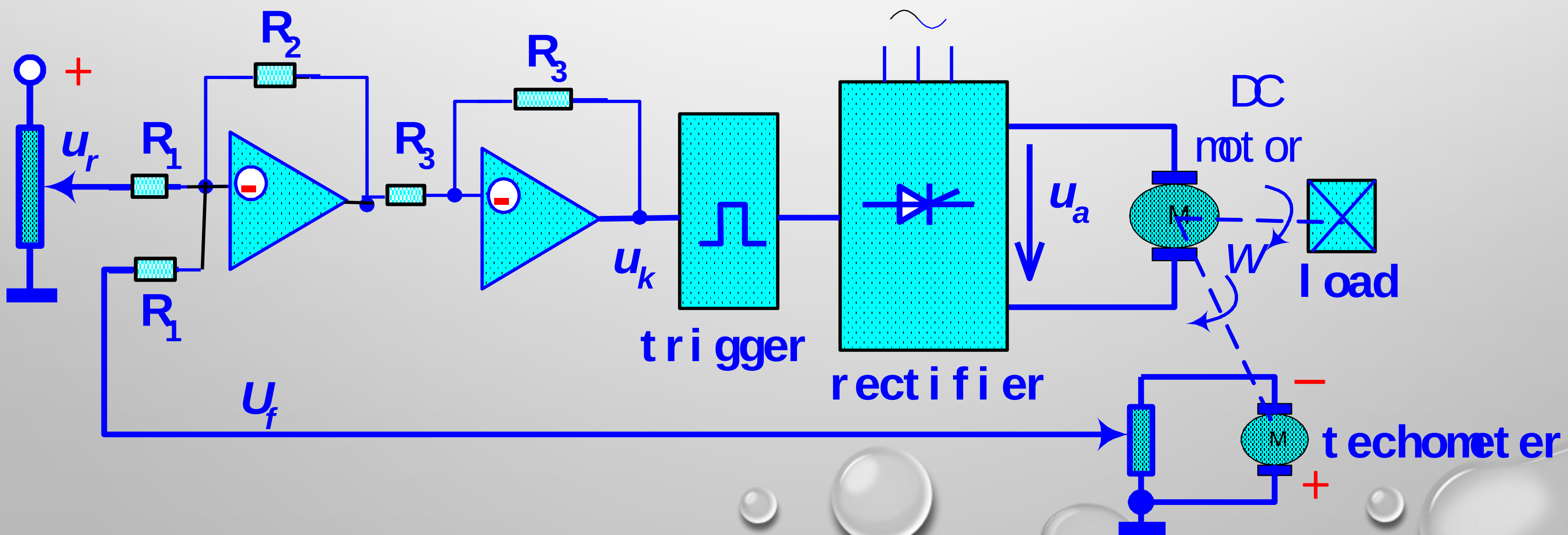
$$=-k\left(1+\frac{1}{\tau s}\right)..................PI\text{-}Controller$$

$$\text{here}:\quad k={R_2}/{R_1}\ ;\quad \tau=R_2C......Integral\ time\ constant$$

5) FOR A CONTROL SYSTEM

- Write the differential equations of the control system;
- Make Laplace transformation, assume zero initial conditions, transform the differential equations into the relevant algebraic equations;
- Deduce: *G(s)=C(s)/R(s)*.

Example 2.13 the DC-Motor control system in Example 2.5



IN EXAMPLE 2.5, WE HAVE WRITTEN DOWN THE DIFFERENTIAL EQUATIONS AS:

$$u_k = \frac{R_2}{R_1}(u_r - u_f) = k_1(u_r - u_f) \ldots\ldots(1)$$

$$u_f = \alpha\omega \ldots\ldots(2) \qquad u_a = k_2 u_k \ldots\ldots(3)$$

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \omega = \frac{1}{C_e} u_a - \frac{T_m}{J}(T_e \dot{\overline{M}} + \overline{M}) \ldots\ldots(4)$$

Make Laplace transformation, we have:

$$U_k(s) = k_1[U_r(s) - U_f(s)] \ldots\ldots(1)$$

$$U_f(s) = \alpha\Omega(s) \ldots\ldots(2) \qquad U_a(s) = k_2 U_k(s) \ldots\ldots(3)$$

$$(T_e T_m s^2 + T_m s + 1)\Omega(s) = \frac{1}{Ce} U_a(s) - \frac{T_e T_m s + T_m}{J} \overline{M}(s) \ldots\ldots(4)$$

(2)→(1)→(3)→(4), we have:

$$[T_eT_ms^2+T_ms+(1+k_1k_2\alpha\frac{1}{C_e})]\Omega(s)=k_1k_2\frac{1}{C_e}U_r(s)-\frac{T_eT_ms+T_m}{J}\overline{M}(s)$$

$$G(s)=\frac{\Omega(s)}{U_r(s)}=\frac{k_1k_2\ {}^{1}\!/\!{}_{C_e}}{T_eT_ms^2+T_ms+(1+k_1k_2\alpha\frac{1}{C_e})}$$

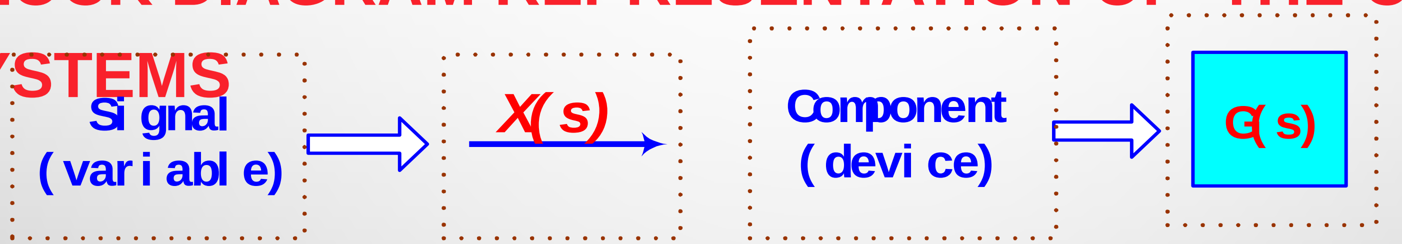
$$here:\quad T_e=\frac{L_a}{R_a}\text{..........}electric-magnetic\ \ time\text{-}constant$$

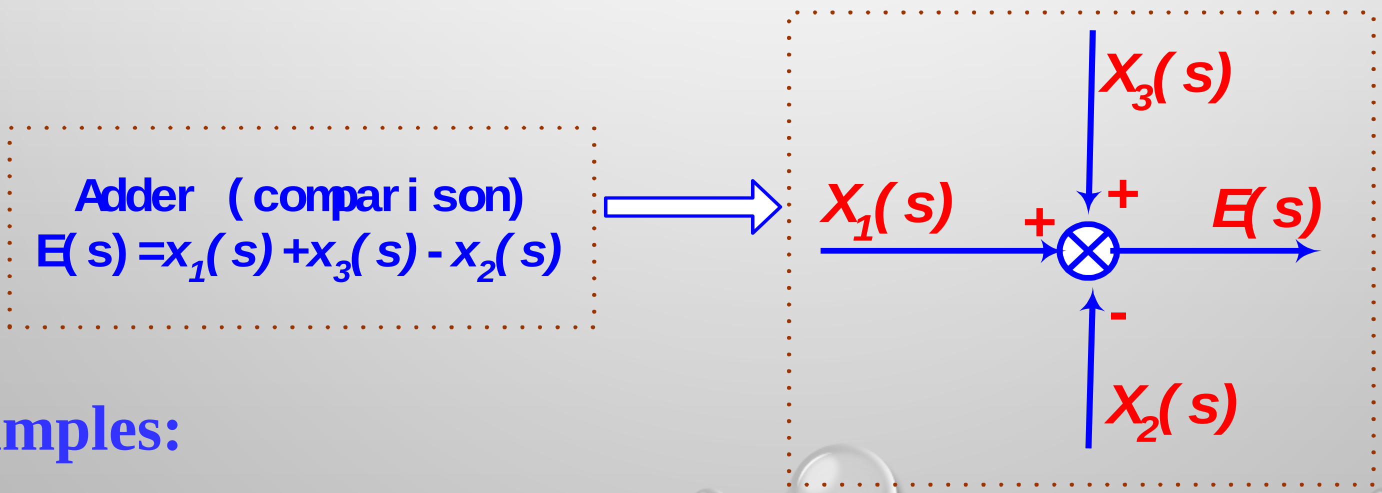
$$T_m=\frac{R_aJ}{C_eC_m}\text{......}mechanical-electric\ \ time\text{-}constant$$

block diagram models (dynamic)

Portray the control systems by the block diagram models more intuitively than the transfer function or differential equation models

BLOCK DIAGRAM REPRESENTATION OF THE CONTROL SYSTEMS





Examples:

EXAMPLE For the DC motor in Example 2.4

In Example 2.4, we have written down the differential equations as:

$$L_a \frac{di_a}{dt} + R_a i_a + E_a = u_a \text{....(1)} \qquad M = C_m i_a \text{........................(2)}$$

$$E_a = C_e \omega \text{........................(3)} \qquad M - \overline{M} = \overline{J}\frac{d\omega}{dt} + \overline{f}\omega \text{.....(4)}$$

Make Laplace transformation, we have:

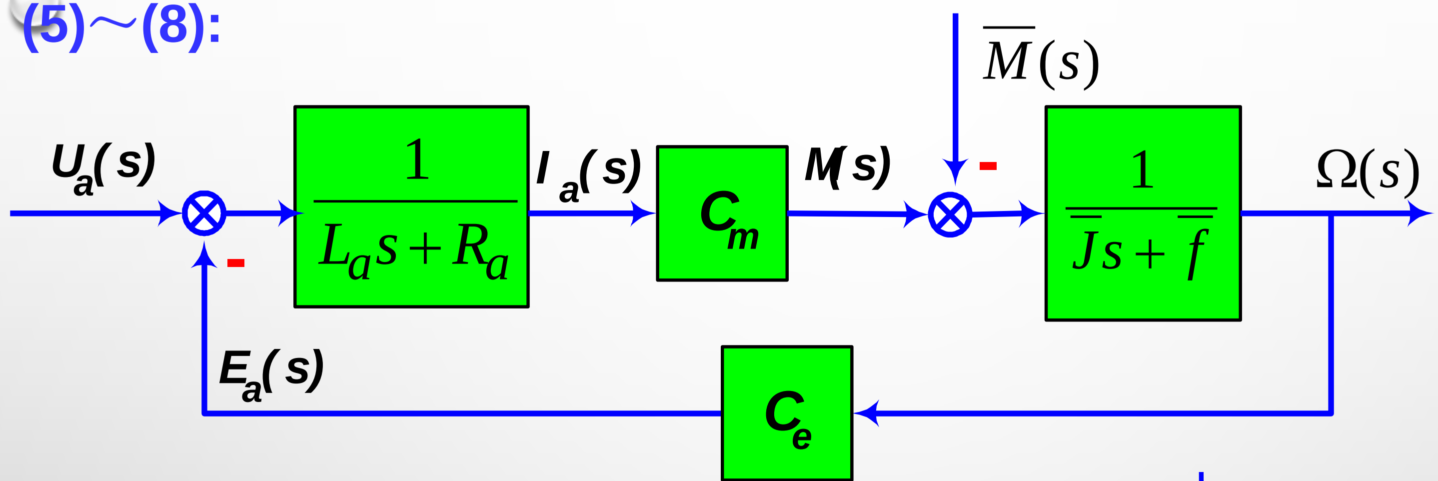
$$L_a s I_a(s) + R_a I_a(s) + E_a(s) = U_a(s) \;\Rightarrow\; I_a(s) = \frac{U_a(s) - E_a(s)}{L_a s + R_a} \text{.............(5)}$$

$$M(s) = C_m I_a(s) \text{...(6)}$$

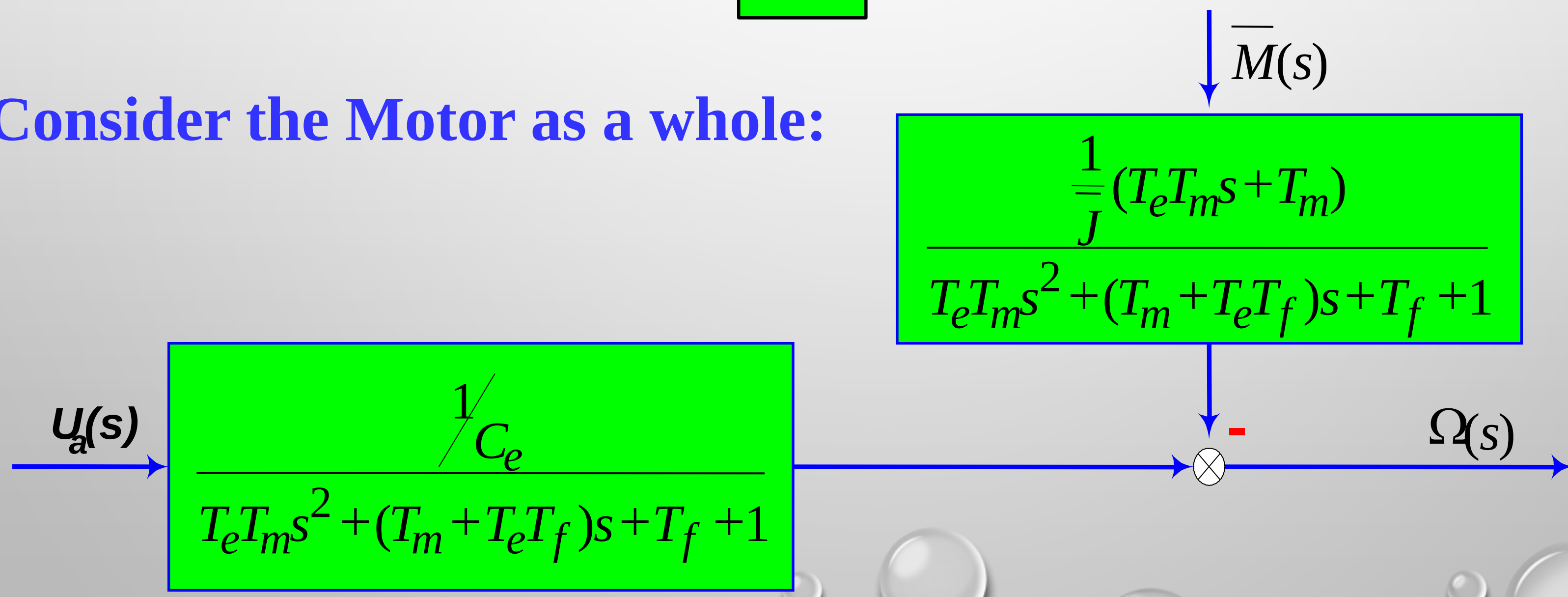
$$E_a(s) = C_e \Omega(s) \text{...(7)}$$

$$M(s) - \overline{M}(s) = \overline{J} s \Omega(s) + \overline{f}\Omega(s) \;\Rightarrow\; \Omega(s) = \frac{1}{\overline{J}s + \overline{f}}[M(s) - \overline{M}(s)] \text{......(8)}$$

DRAW BLOCK DIAGRAM IN TERMS OF THE EQUATIONS (5)～(8):

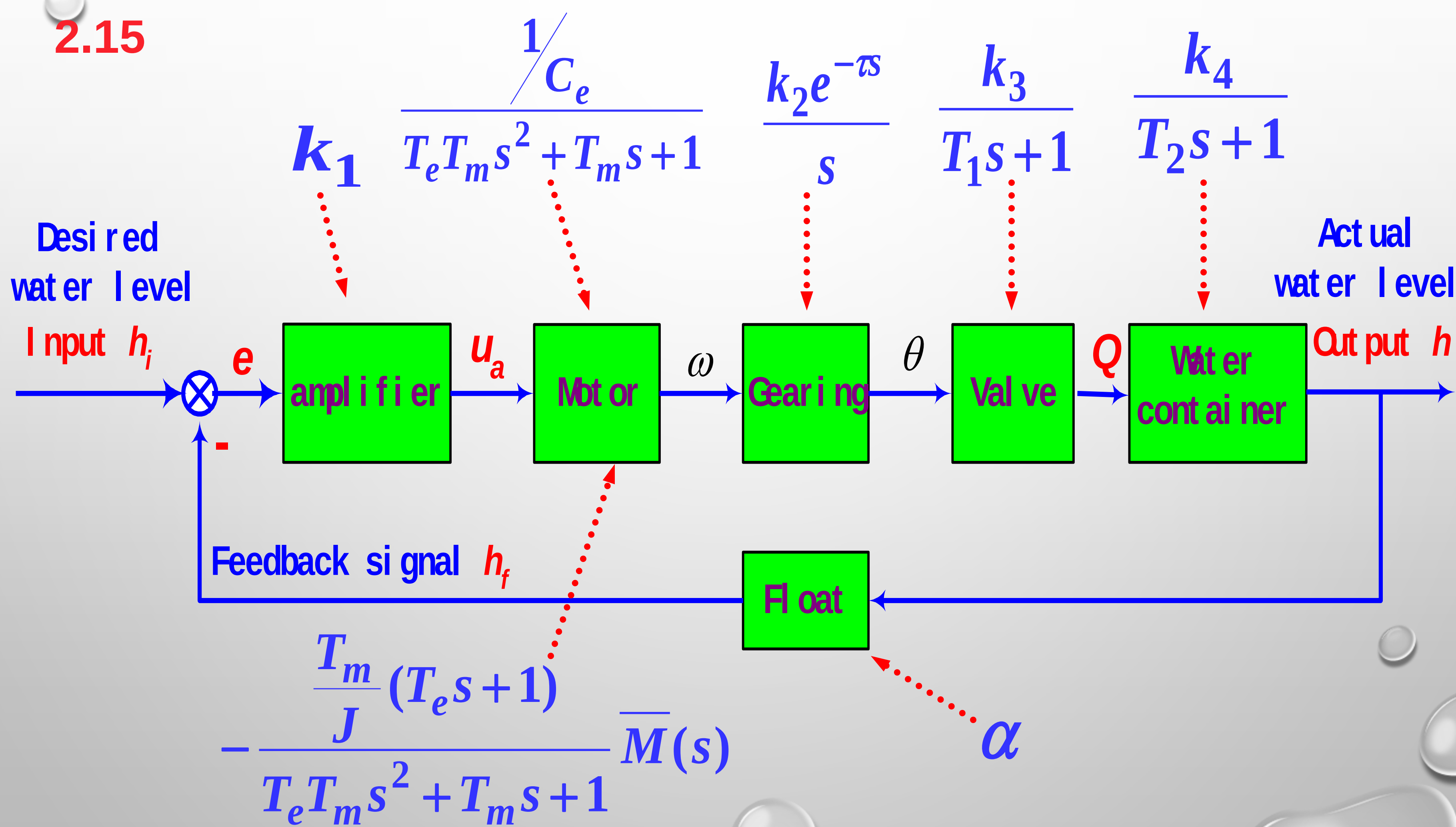


Consider the Motor as a whole:

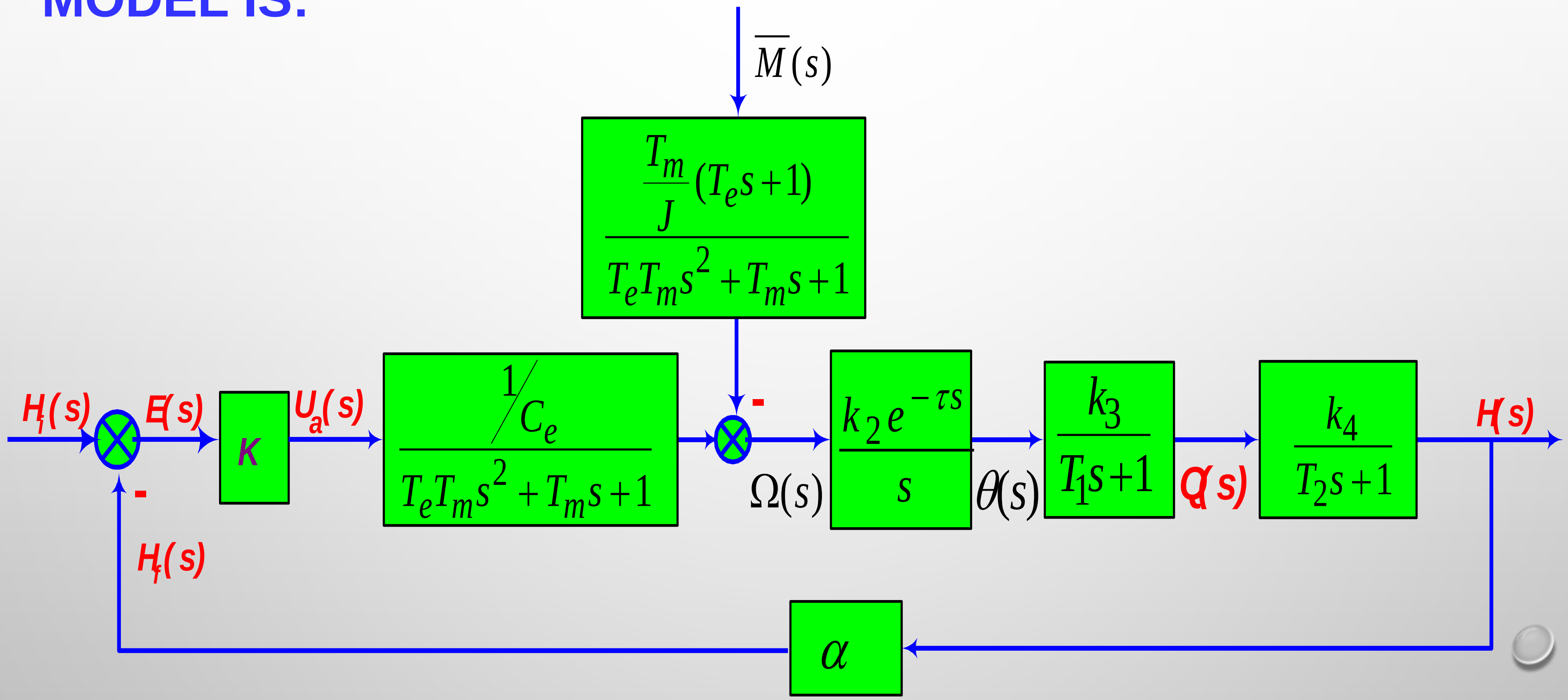


EXAMPLE 2.15

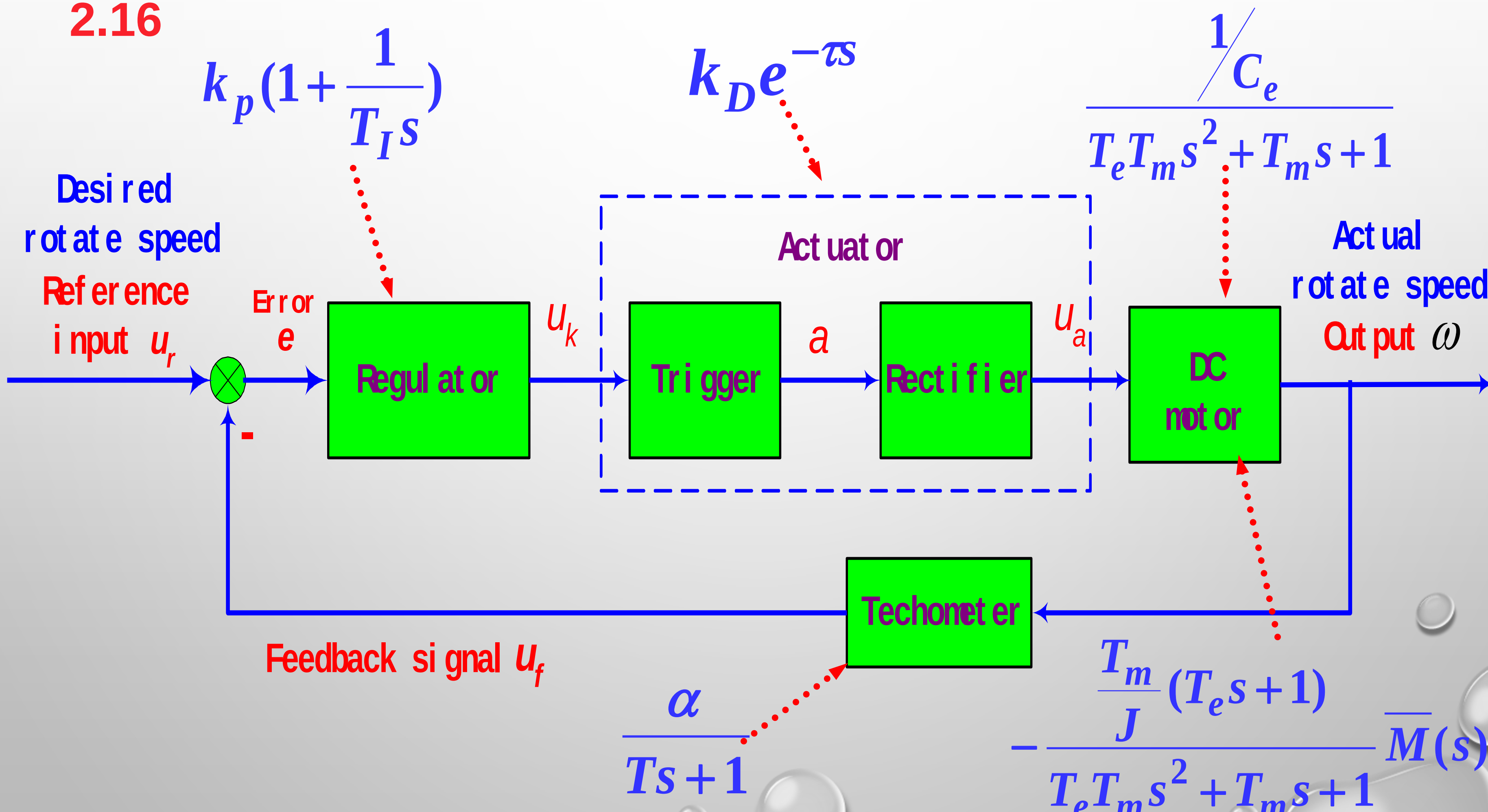
The water level control system in Fig 1.8:



THE BLOCK DIAGRAM MODEL IS:



EXAMPLE 2.16 — **The DC motor control system in Fig 1.9**



THE BLOCK DIAGRAM MODEL IS: