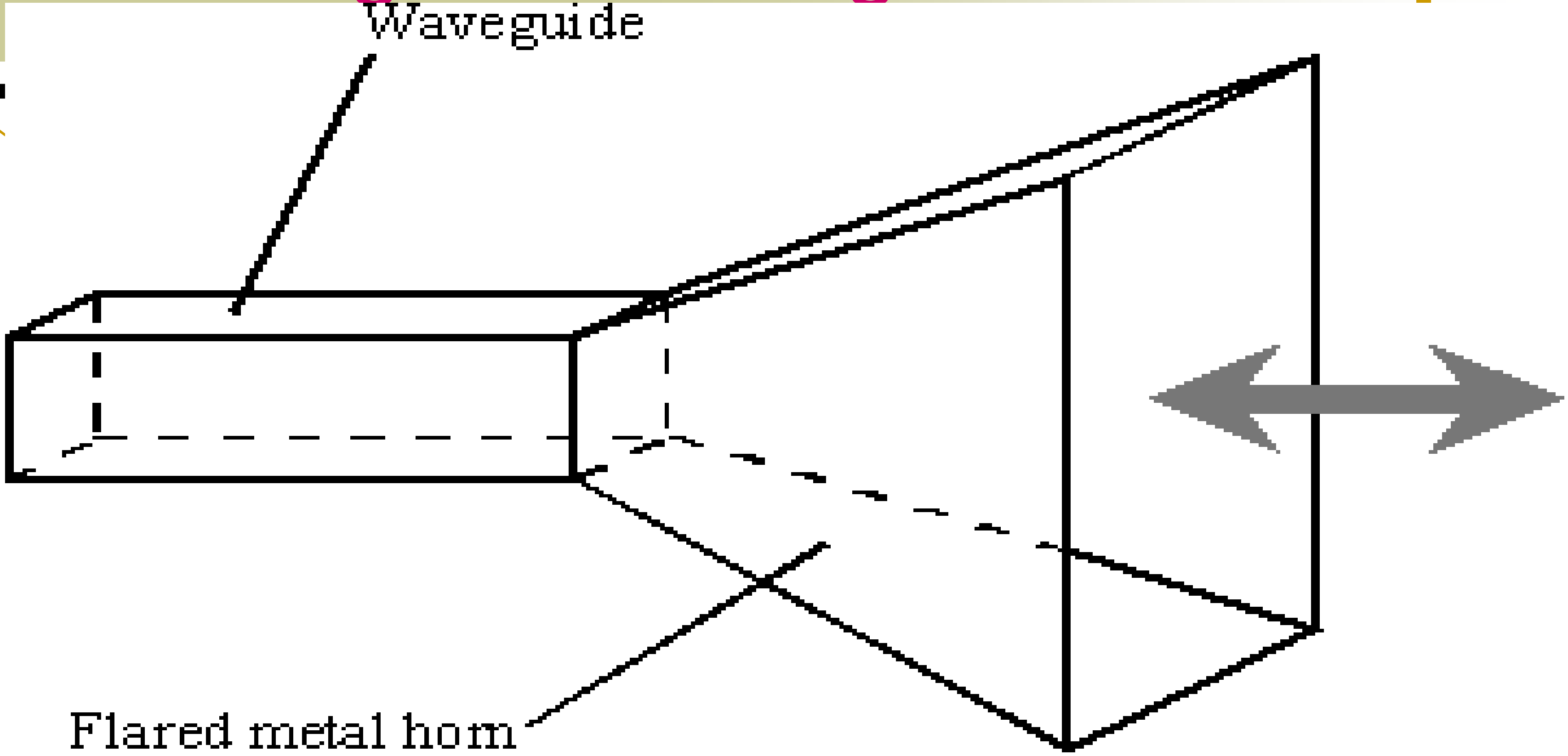
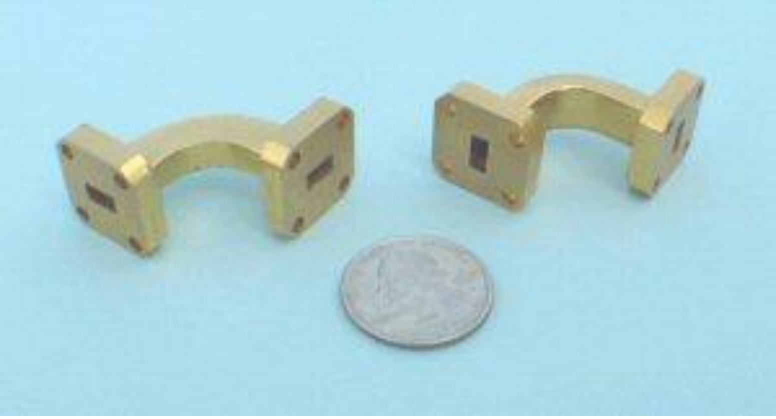
Rectangular Waveguides



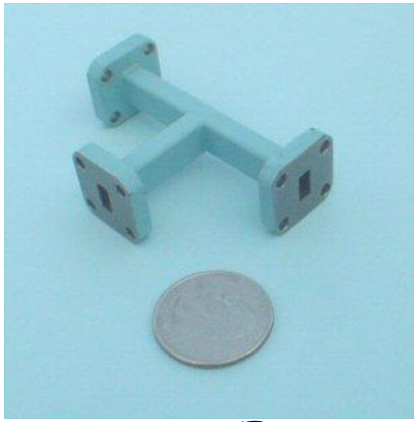
Waveguide components

Rectangular waveguide

Waveguide to coax adapter



Waveguide bends



E-tee

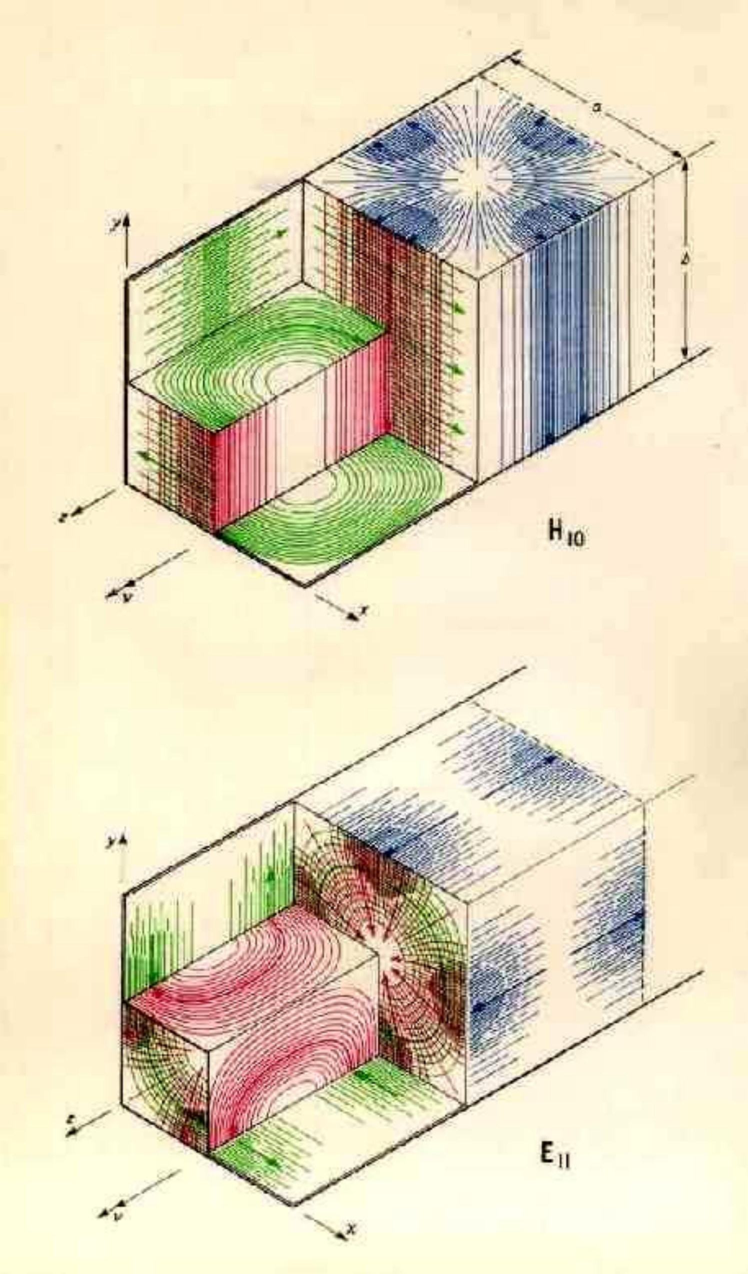
More waveguides

Uses

- To reduce attenuation loss
  - High frequencies
  - High power
- Can operate only above certain frequencies
  - Acts as a High-pass filter
- Normally circular or <u>rectangular</u>
  - We will assume lossless rectangular

Rectangular WG

- Need to <u>find the fields components</u> of the *em* wave inside the waveguide
  - $E_z\ H_z\ E_x\ H_x\ E_y\ H_y$
- We'll find that *waveguides* <u>don't support TEM</u> waves



Rectangular Waveguides: *Fields inside*

Using phasors & assuming waveguide filled with

- lossless dielectric material and
- walls of perfect conductor,

the wave inside should obey…

$$\nabla^2 E + k^2 E = 0$$

$$\nabla^2 H + k^2 H = 0$$

$$\text{where} \quad k^2 = \omega^2 \mu \varepsilon_c$$

Then applying on the z-component…

$$\nabla^2 E_z + k^2 E_z = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0$$

Solving by method of Separation of Variables :

$$E_z(x, y, z) = X(x)Y(y)Z(z)$$

from where we obtain :

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2$$

Fields inside the waveguide

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=-k^2$$

$$-k_x^2-k_y^2+\gamma^2=-k^2 \qquad h^2=\gamma^2+k^2=k_x^2+k_y^2$$

which results in the expressions:

$$X''+k_x^2X=0 \qquad X(x)=c_1\cos k_xx+c_2\sin k_xx$$

$$Y''+k_y^2Y=0 \qquad Y(y)=c_3\cos k_yy+c_4\sin k_yy$$

$$Z''-\gamma^2Z=0 \qquad Z(z)=c_5e^{\gamma z}+c_6e^{-\gamma z}$$

Substituting

$$X(x) = c_1 \cos k_x x + c_2 \sin k_x x$$

$$Y(y) = c_3 \cos k_y y + c_4 \sin k_y y$$

$$E_z(x, y, z) = X(x)Y(y)Z(z)$$

$$Z(z) = c_5 e^{\gamma z} + c_6 e^{-\gamma z}$$

$$E_z = (c_1 \cos k_x x + c_2 \sin k_x x)(c_3 \cos k_y y + c_4 \sin k_y y)(c_5 e^{\gamma z} + c_6 e^{-\gamma z})$$

If only looking at the wave traveling in $+z$-direction:

$$E_z = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y)e^{-\gamma z}$$

Similarly for the magnetic field,

$$H_z = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)e^{-\gamma z}$$

Other components

From Faraday and Ampere Laws we can find the remaining four components:

$$E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$H_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

where

$$h^2 = \gamma^2 + k^2 = k_x^2 + k_y^2$$

*So once we know E_z and H_z, we can find all the other fields.

Modes of propagation

From these equations we can conclude:

- TEM ($E_z = H_z = 0$) can't propagate.
- TE ($E_z = 0$) transverse electric
  - In TE mode, the electric lines of flux are perpendicular to the axis of the waveguide
- TM ($H_z = 0$) transverse magnetic, E_z exists
  - In TM mode, the magnetic lines of flux are perpendicular to the axis of the waveguide.
- HE hybrid modes in which all components exists

TM Mode

$$E_z = \left(A_1 \cos k_x x + A_2 \sin k_x x\right)\left(A_3 \cos k_y y + A_4 \sin k_y y\right) e^{-\gamma z}$$

- Boundary conditions: $E_z = 0 \text{ at } y = 0, b$

  $E_z = 0 \text{ at } x = 0, a$

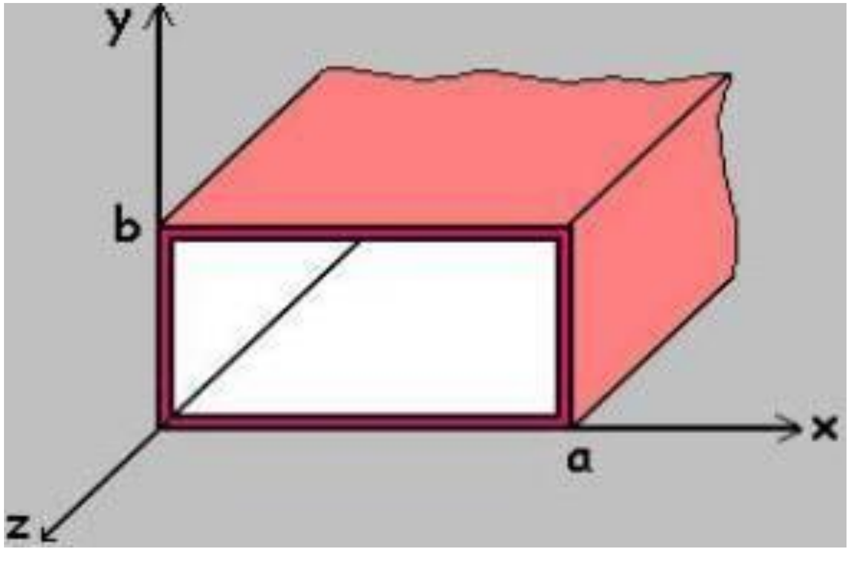
From these, we conclude:

X(x) is in the form of **sin $k_x x$**, where $k_x = m\pi/a$, $m = 1,2,3,\ldots$

Y(y) is in the form of **sin $k_y y$**, where $k_y = n\pi/b$, $n = 1,2,3,\ldots$

So the solution for $E_z(x,y,z)$ is

$$E_z = A_2 A_4 \left(\sin k_x x\right)\left(\sin k_y y\right) e^{-j\beta z}$$



TM Mode

- Substituting

$$E_z = E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

where

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad = \gamma^2 + k^2$$

TM_{mn}

$$E_z = E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_z = 0$$

- Other components are

$$E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \qquad E_x = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} \qquad E_y = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} \qquad H_x = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} \qquad H_y = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

TM modes

- The *m* and *n* represent the mode of propagation and indicates the number of variations of the field in the *x* and *y* directions
- Note that for the TM mode, if *n* or *m* is zero, all fields are zero.

TM Cutoff

$$\gamma = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}$$
$$= \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}$$

- The cutoff frequency occurs when

$$\text{When } \omega_c^{\,2}\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \text{then } \gamma = \alpha + j\beta = 0$$

$$\text{or } f_c = \frac{1}{2\pi}\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

- Evanescent:

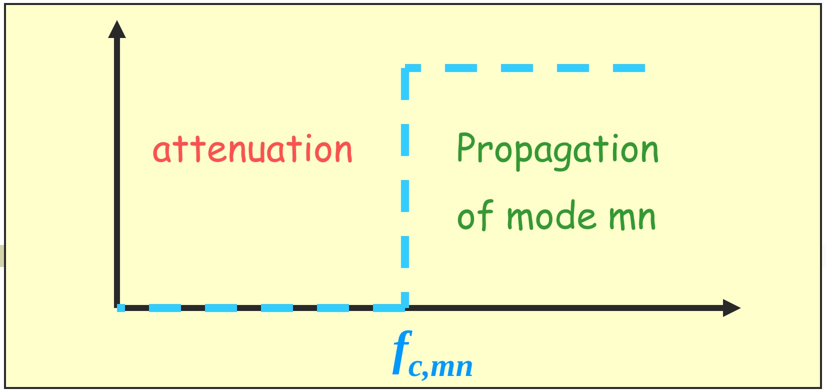
$$\text{When } \omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \gamma = \alpha \text{ and } \beta = 0$$

  - Means no propagation, everything is attenuated

- Propagation: $$\text{When } \omega^2\mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \gamma = j\beta \text{ and } \alpha = 0$$

  - This is the case we are interested since is when the wave is allowed to travel through the guide.

Cutoff



- The cutoff frequency is the frequency below which attenuation occurs and above which propagation takes place. (High Pass)

$$f_{c\,mn} = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- The phase constant becomes

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \beta'\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Phase velocity and impedance

- The phase velocity is defined as

$$u_p = \frac{\omega}{\beta'} \qquad \lambda = \frac{2\pi}{\beta} = \frac{u_p}{f}$$

- And the intrinsic impedance of the mode is

$$\eta_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}$$

Summary of TM modes

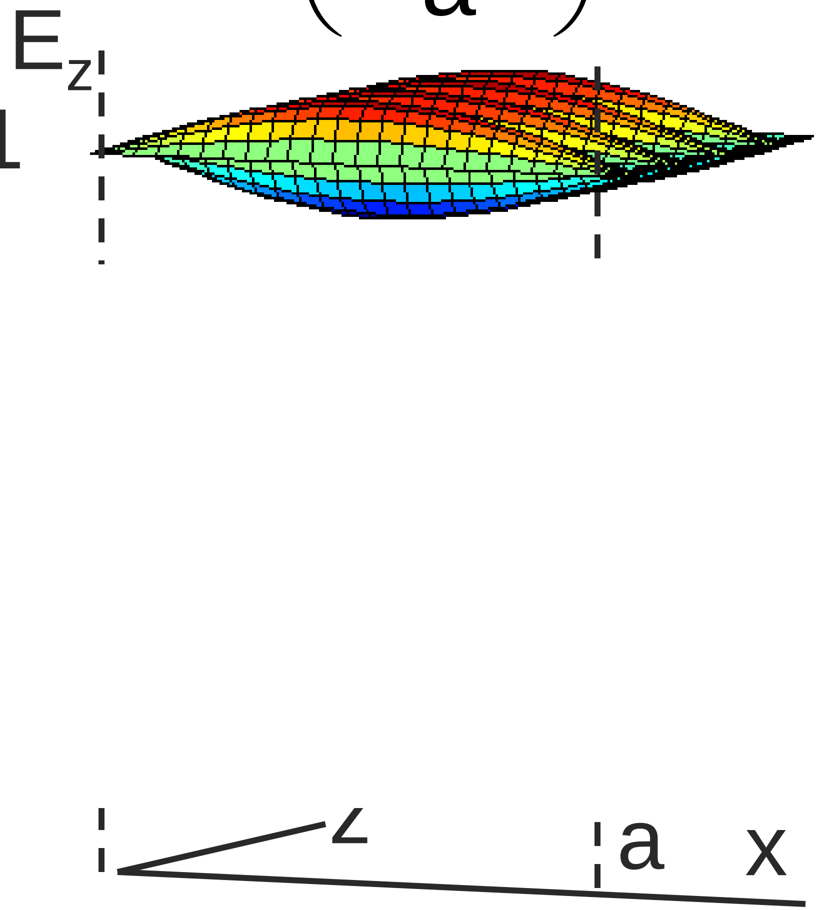
Wave in the dielectric medium	Inside the waveguide
$\beta' = \omega / u' = \omega\sqrt{\mu\varepsilon}$	$\beta = \beta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}$
$\eta' = \sqrt{\mu / \varepsilon}$	$\eta_{TM} = \eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}$
$u' = \omega / \beta' = f\lambda = 1 / \sqrt{\mu\varepsilon}$	$u_p = \frac{\omega}{\beta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}} = \omega / \beta$
$\lambda' = u' / f$	$\lambda = \frac{\lambda'}{\sqrt{1 - \left[\frac{f_c}{f}\right]^2}}$

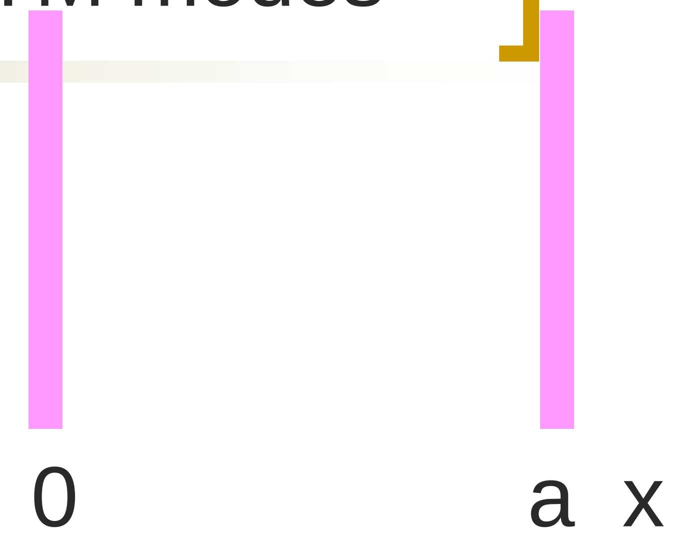
Related example of how fields look:

Parallel plate waveguide - TM modes

$$E_z = A \sin\left(\frac{m\pi x}{a}\right) e^{j(\omega t - \beta z)}$$

m = 1





TE Mode

$$H_z = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)e^{-\gamma z}$$

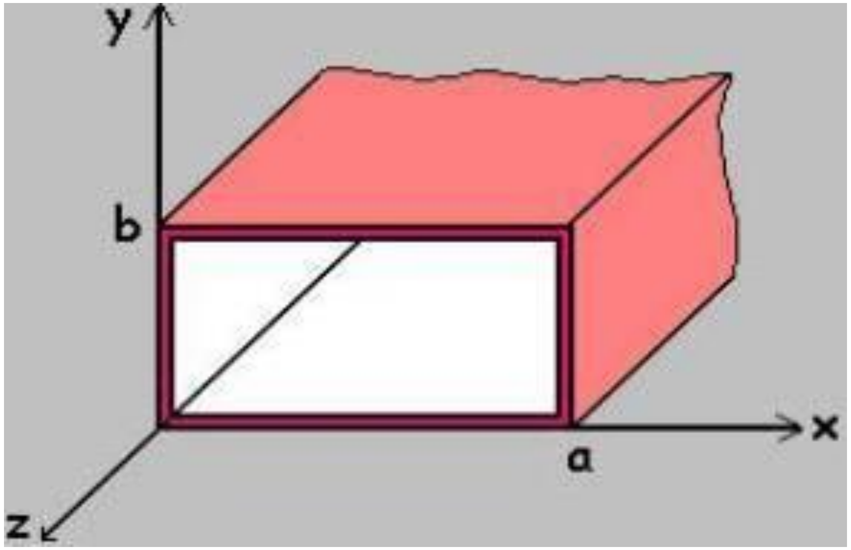
- Boundary conditions: $E_x = 0 \text{ at } y = 0, b$
  $E_y = 0 \text{ at } x = 0, a$

From these, we conclude:

X(x) is in the form of **cos $k_x x$**,
where $k_x = m\pi/a$, m=0,1,2,3,…

Y(y) is in the form of **cos $k_y y$**,
where $k_y = n\pi/b$, n=0,1,2,3,…

So the solution for $E_z(x,y,z)$ is



$$H_z = B_1 B_3 (\cos k_x x)(\cos k_y y)e^{-j\beta z}$$

TE Mode

- Substituting

$$H_z = H_o \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

where again

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

- Note that <u>*n* and *m* cannot be both zero</u> because the fields will all be zero.

TE_{mn}

$$H_z = H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_z = 0$$

- Other components are

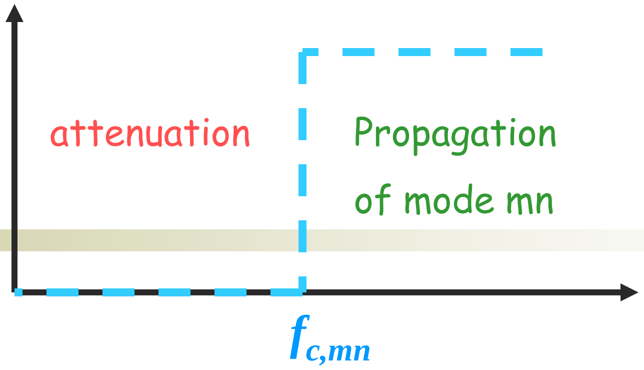
$$E_x = -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \qquad E_x = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$E_y = -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \qquad E_y = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} \qquad H_x = \frac{j\beta}{h^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y} \qquad H_y = \frac{j\beta}{h^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

Cutoff



- The cutoff frequency is the same expression as for the TM mode

$$f_{c\,mn} = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- But the lowest attainable frequencies are lowest because here n or m can be zero.

Dominant Mode

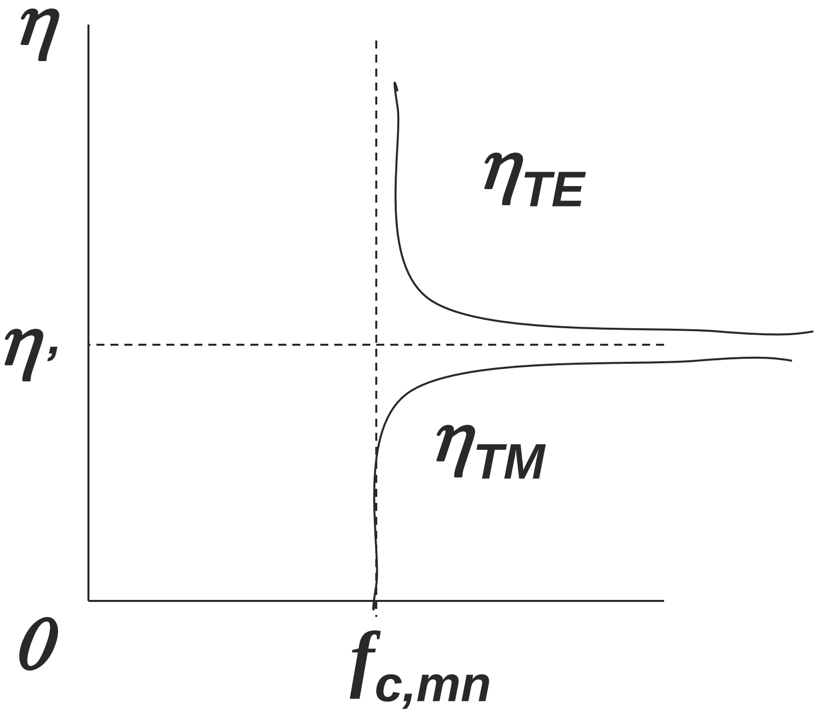
- The dominant mode is the mode with lowest cutoff frequency.
- It’s always TE_{10}
- The order of the next modes change depending on the dimensions of the guide.

Summary of TE modes

Wave in the dielectric medium	Inside the waveguide
$\beta' = \omega / u' = \omega\sqrt{\mu\varepsilon}$	$\beta = \beta'\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$\eta' = \sqrt{\mu / \varepsilon}$	$\eta_{TE} = \frac{\eta'}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$u' = \omega / \beta' = f\lambda = 1/\sqrt{\mu\varepsilon}$	$u_p = \frac{\omega}{\beta'\sqrt{1-\left[\frac{f_c}{f}\right]^2}} = \omega / \beta$
$\lambda' = u' / f$	$\lambda = \frac{\lambda'}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$

Variation of wave impedance

- Wave impedance varies with frequency and mode



Example:

Consider a length of air-filled copper X-band waveguide, with dimensions a=2.286cm, b=1.016cm operating at 10GHz. Find the cutoff frequencies of all possible propagating modes.

Solution:

- From the formula for the cut-off frequency

$$f_{c_{mn}} = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Example

An air-filled 5-by 2-cm waveguide has

$$E_z = 20\sin(40\pi x)\sin(50\pi y)e^{-j\beta z} \text{ V/m}$$

at 15GHz

- What mode is being propagated?
- Find β
- Determine E_y/E_x

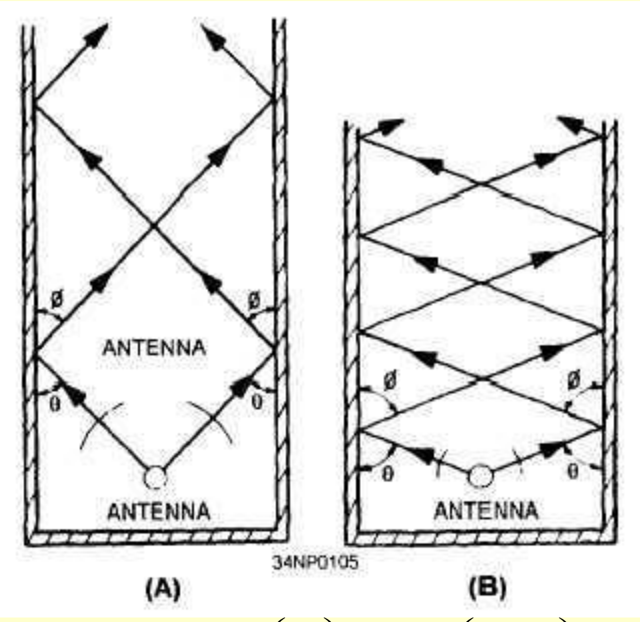
Group velocity, u_g

- Is the velocity at which the energy travels.

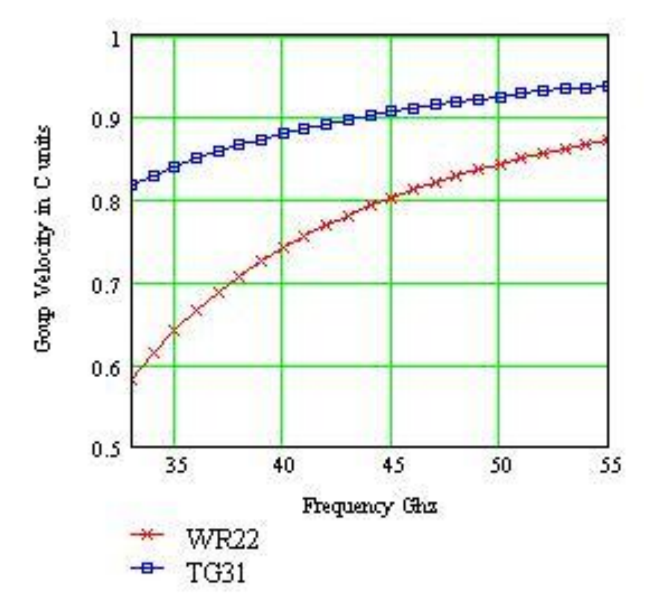
$$u_g = \frac{1}{\partial\beta / \partial\omega} = u' \sqrt{1 - \left[\frac{f_c}{f}\right]^2} \quad \left[\frac{\text{rad/s}}{\text{rad/m}}\right] = \left[\frac{m}{s}\right]$$

- It is always less than u'

$$\boxed{u_p u_g = (u')^2}$$

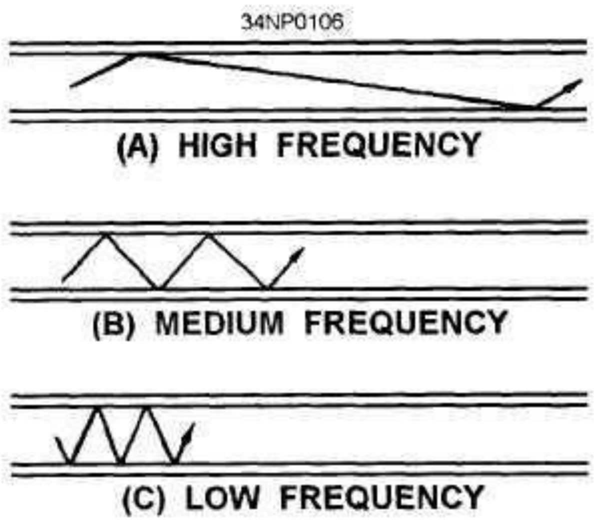


$$E_y = -\frac{j\omega\mu}{h^2}\left(\frac{\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$



Group Velocity

- As frequency is increased, the group velocity increases.



Power transmission

- The average Poynting vector for the waveguide fields is

$$P_{ave} = \frac{1}{2}\mathrm{Re}\left[E \times H^*\right] = \frac{1}{2}\mathrm{Re}\left[E_x H_y{}^* - E_y H_x{}^*\right]$$

$$= \frac{\left|E_x\right|^2 + \left|E_y\right|^2}{2\eta}\hat{z} \qquad \textbf{[W/m}^2\textbf{]}$$

- where $\eta = \eta_{TE}$ or η_{TM} depending on the mode

$$P_{ave} = \int P_{ave} \cdot dS = \int_{x=0}^{a}\int_{y=0}^{b} \frac{\left|E_x\right|^2 + \left|E_y\right|^2}{2\eta} dy dx \qquad \textbf{[W]}$$

Attenuation in Lossy waveguide

- When dielectric inside guide is lossy, and walls are not perfect conductors, power is lost as it travels along guide.

$$P_{ave} = P_o e^{-2\alpha z}$$

- The loss power is $P_L = -\frac{dP_{ave}}{dz} = 2\alpha P_{ave}$
- Where $\alpha = \alpha_c + \alpha_d$ are the attenuation due to ohmic (conduction) and dielectric losses
- Usually $\alpha_c >> \alpha_d$

Attenuation for TE_{10}

- Dielectric attenuation, Np/m

$$\alpha_d = -\frac{\sigma\eta'}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

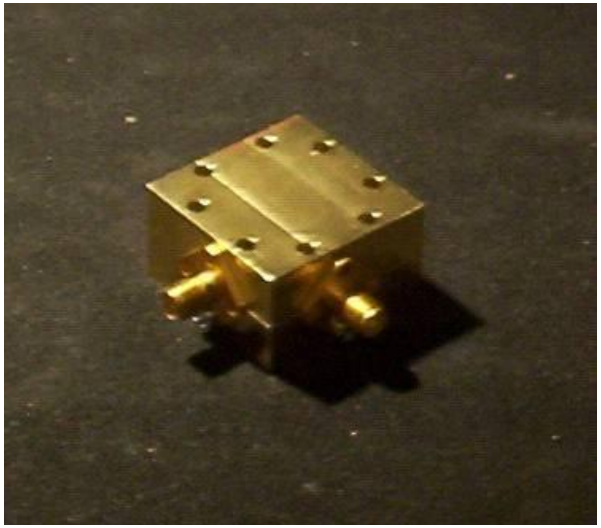
Dielectric conductivity!

- Conductor attenuation, Np/m

$$\alpha_c = -\frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_{c,10}}{f}\right)^2}}\left(0.5+\frac{b}{a}\left(\frac{f_{c,10}}{f}\right)^2\right)$$

Waveguide Cavities

- Cavities, or resonators, are used for storing energy
- Used in klystron tubes, band-pass filters and frequency meters
- It's equivalent to an RLC circuit at high frequency
- Their shape is that of a cavity, either cylindrical or cubical.



Cavity TM Mode to z

Solving by Separation of Variables :

$$E_z(x, y, z) = X(x)Y(y)Z(z)$$

from where we obtain :

$$X(x) = c_1 \cos k_x x + c_2 \sin k_x x$$

$$Y(y) = c_3 \cos k_y y + c_4 \sin k_y y$$

$$Z(z) = c_5 \cos k_z z + c_6 \sin k_z z$$

$$where \quad k^2 = k_x^2 + k_y^2 + k_z{}^2$$

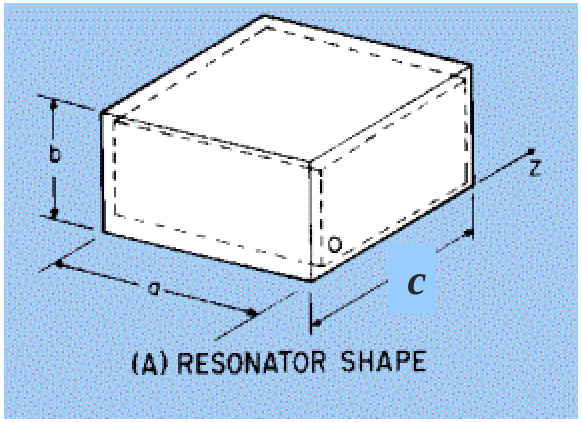
TM$_{mnp}$ Boundary Conditions

$$E_z = 0 \text{ at } y = 0, b$$
$$E_z = 0 \text{ at } x = 0, a$$
$$E_y = E_x = 0, \text{ at } z = 0, c$$

From these, we conclude:

$k_x = m\pi/a$
$k_y = n\pi/b$
$k_z = p\pi/c$

where c is the dimension in z-axis



$$E_z = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{c}\right)$$

where

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2 = \omega^2\mu\varepsilon$$

Resonant frequency

- The resonant frequency is the same for TM or TE modes, except that the lowest-order TM is TM_{111} and the lowest-order in TE is TE_{101}.

$$f_r = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

Cavity TE Mode to z

Solving by Separation of Variables :

$$H_z(x, y, z) = X(x)Y(y)Z(z)$$

from where we obtain :

$$X(x) = c_1 \cos k_x x + c_2 \sin k_x x$$

$$Y(y) = c_3 \cos k_y y + c_4 \sin k_y y$$

$$Z(z) = c_5 \cos k_z z + c_6 \sin k_z z$$

$$where \quad k^2 = k_x^2 + k_y^2 + k_z^{\ 2}$$

TE_{mnp} Boundary Conditions

$$H_z = 0 \text{ at } z = 0, c$$
$$E_y = 0 \text{ at } x = 0, a$$
$$E_x = 0, \text{ at } y = 0, b$$

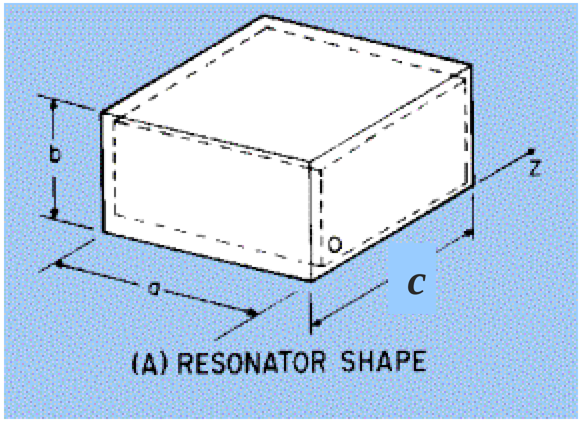
From these, we conclude:

$k_x = m\pi/a$

$k_y = n\pi/b$

$k_z = p\pi/c$

where c is the dimension in z-axis

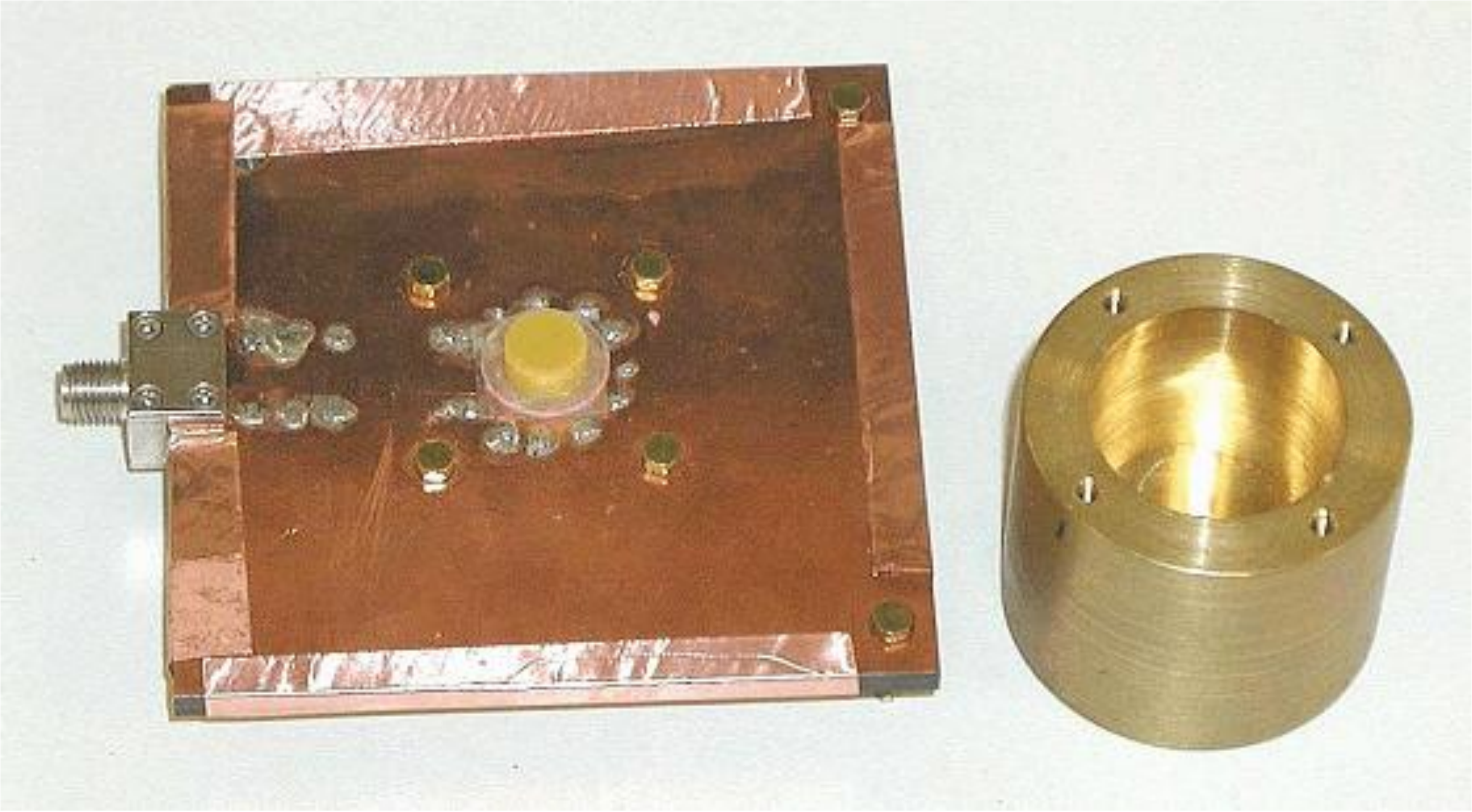


$$H_z = H_o \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi y}{c}\right)$$

Quality Factor, *Q*

- The cavity has walls with finite conductivity and is therefore losing stored energy.
- The quality factor, *Q*, characterized the loss and also the bandwidth of the cavity resonator.
- Dielectric cavities are used for resonators, amplifiers and oscillators at microwave frequencies.

A dielectric resonator antenna with a cap for measuring the radiation efficiency



Quality Factor, *Q*

- Is defined as

$$Q = 2\pi \frac{Time\ average\ energy\ stored}{loss\ energy\ per\ cycle\ of\ oscillation}$$

$$= 2\pi \frac{W}{P_L}$$

For the dominant mode TE_{101}

$$Q_{TE_{101}} = \frac{\left(a^2 + c^2\right)abc}{\delta\left[2b\left(a^3 + c^3\right) + ac\left(a^2 + c^2\right)\right]}$$

where

$$\delta = \frac{1}{\sqrt{\pi f_{101} \mu_o \sigma_c}}$$