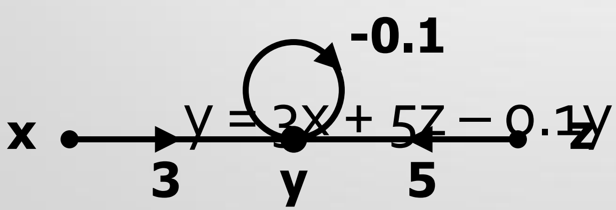
Control Systems

Signal Flow Graph

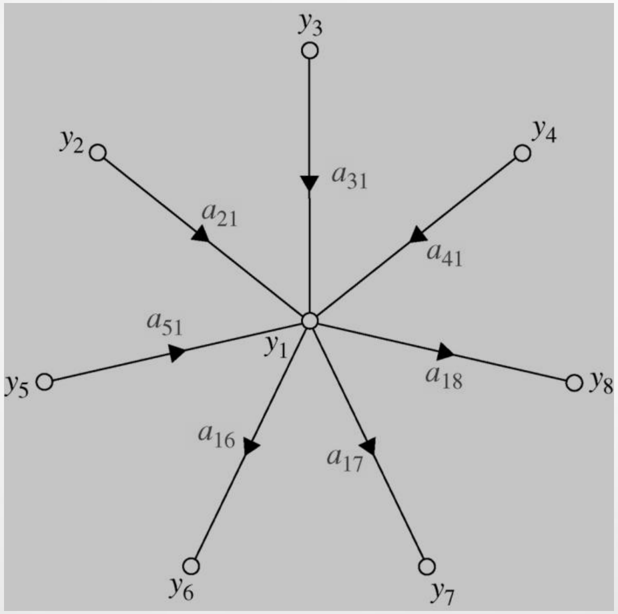
- nodes : variables
- branches : gains

e.g. $y = a \cdot x$

e.g. $y = 3x + 5z - 0.1y$

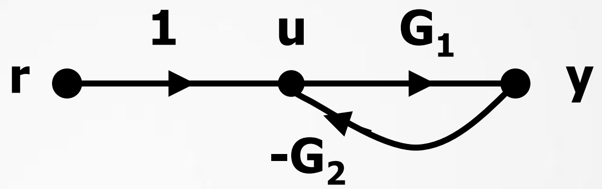


The value of a node is equal to the sum of all signal coming into the node.
The incoming signal needs to be weighted by the branch gains.



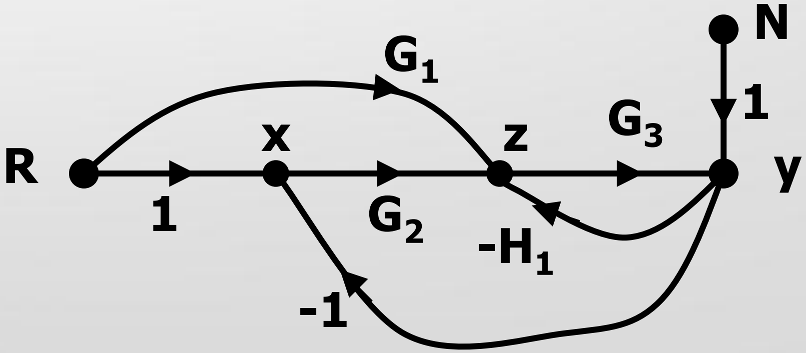
$$y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$$

Note: y6, y7, y8 are gained up versions of y1.

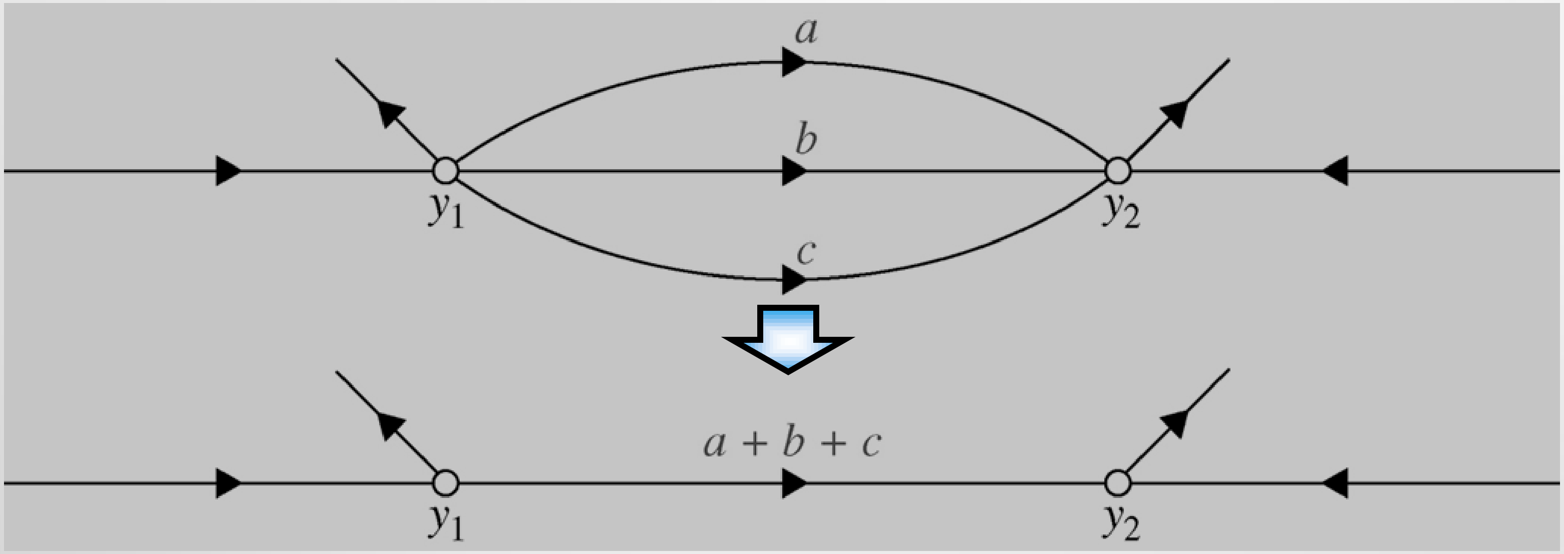


An input node is a node with only out going arrows.

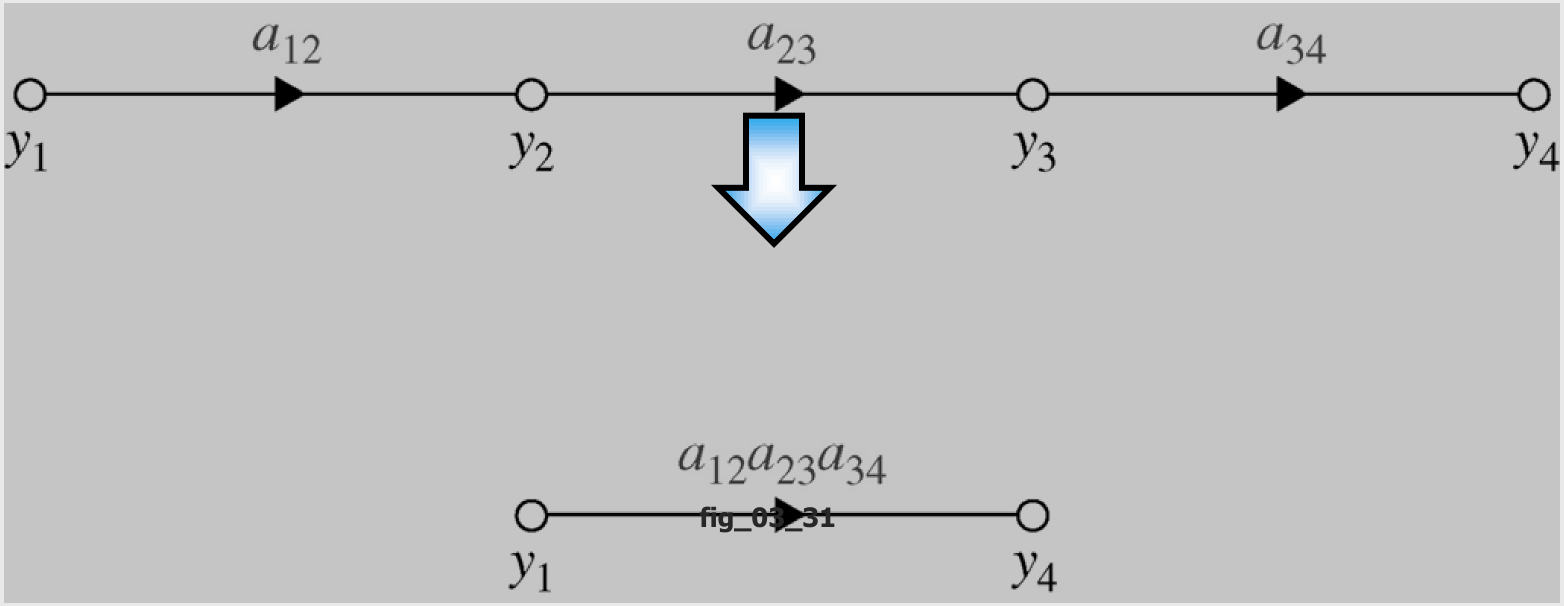
R, N and r are input nodes.



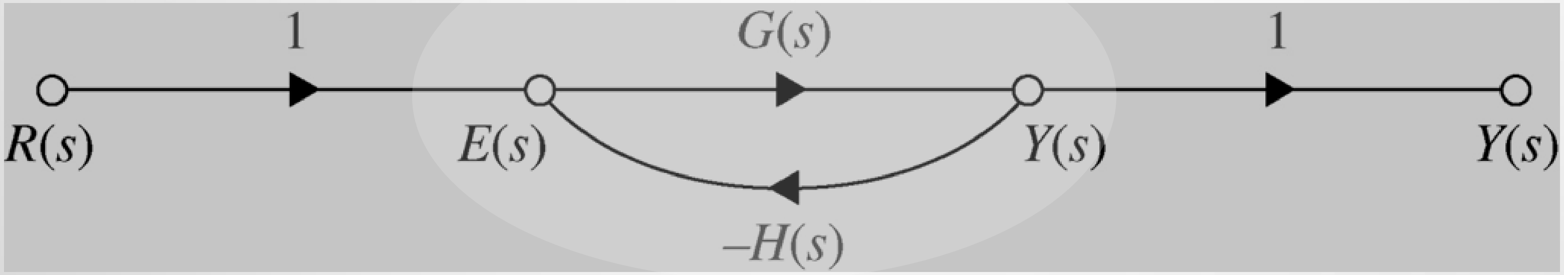
Parallel branches can be summed to form a single branch

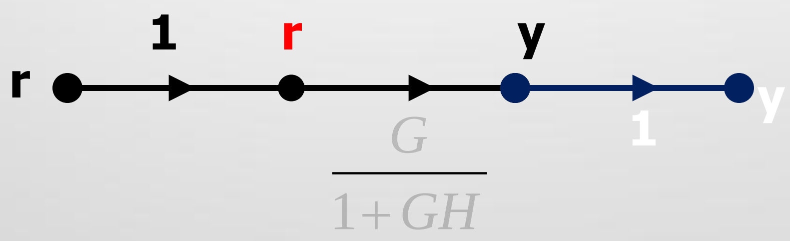


Series branches can be multiplied to form a single branch

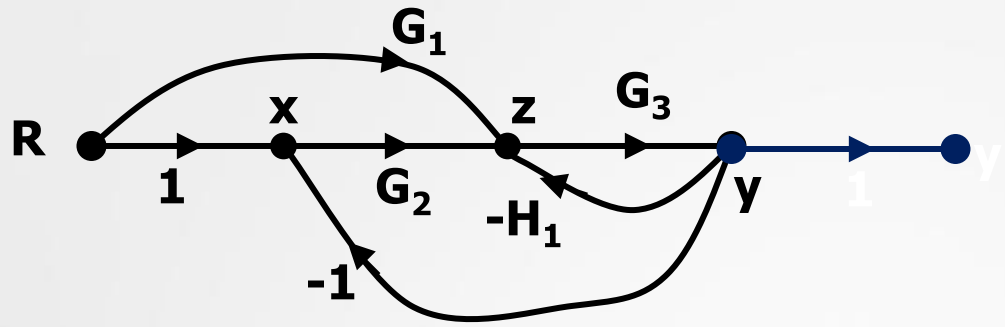


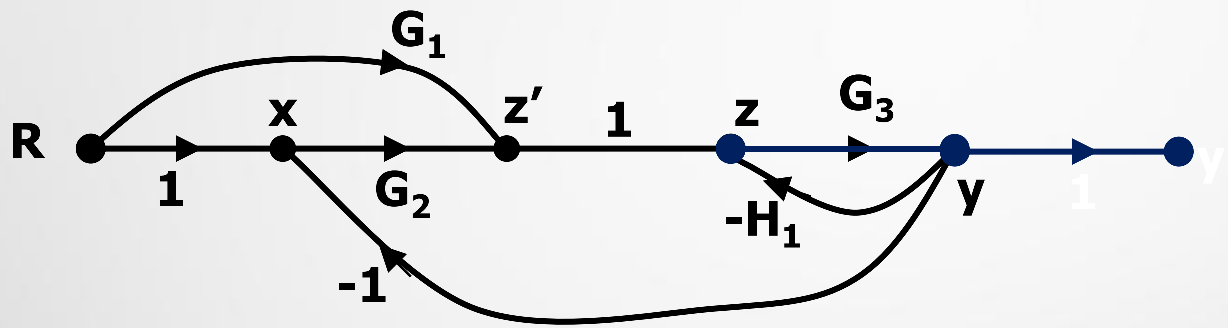
Feedback connections can be simplified into a single branch

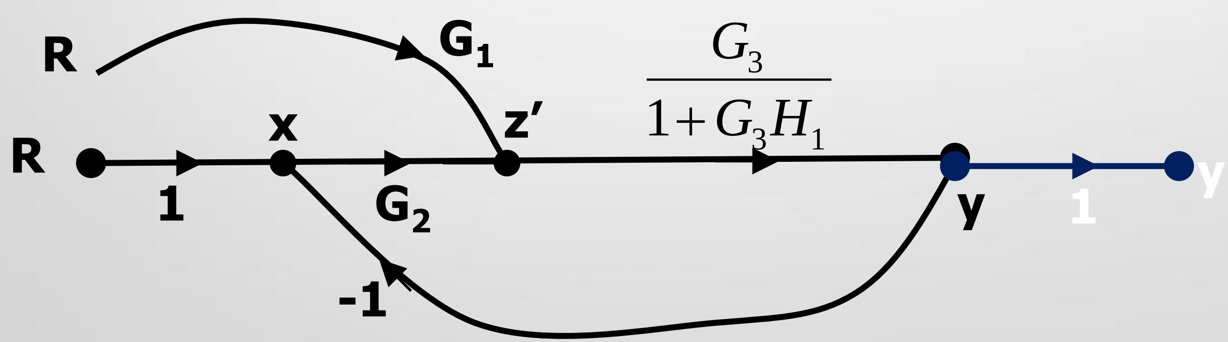


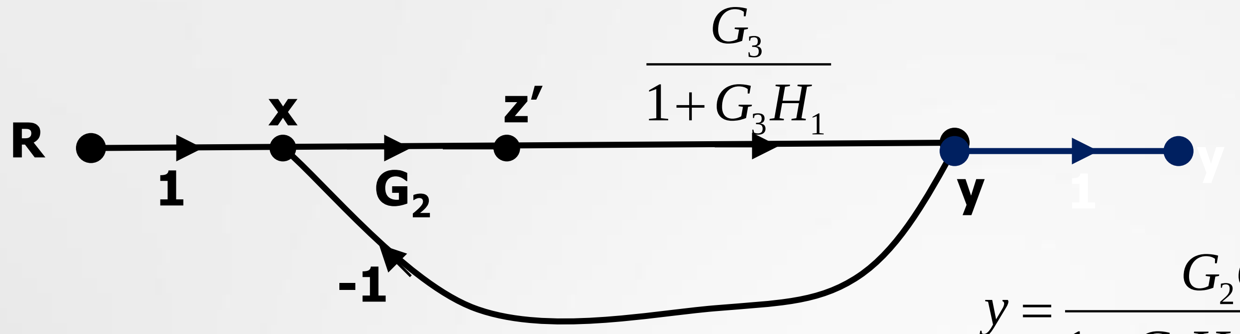


Note: the internal node E is lost!

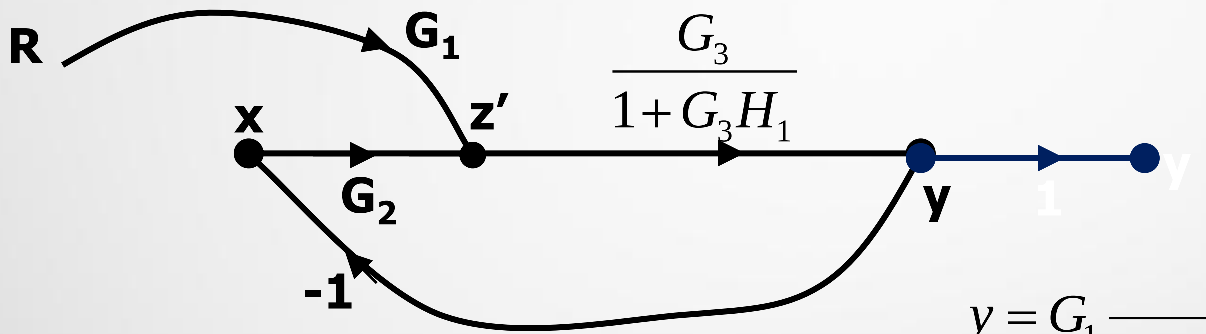

G1
R
x
z
G3
1
G2
y
-H1
-1


G1
R
x
z′
1
z
G3
1
G2
y
-H1
-1


R
G1
$\frac{G_3}{1+G_3H_1}$
R
x
z′
1
G2
y
-1



$$y = \frac{G_2 G_3}{1 + G_3 H_1 + G_2 G_3} R$$



$$y = G_1 \frac{G_3}{1 + G_3 H_1 + G_2 G_3} R$$

Overall:

$$y = \left(\frac{G_2 G_3}{1 + G_3 H_1 + G_2 G_3} + \frac{G_1 G_3}{1 + G_3 H_1 + G_2 G_3} \right) R$$

$$= \frac{G_2 G_3 + G_1 G_3}{1 + G_3 H_1 + G_2 G_3} R$$

Mason's Gain Formula

- A forward path: a path from input to output
- Forward path gain M_k: total product of gains along the path
- A loop is a closed path in which you can start at any point, follow the arrows, and come back to the same point
- A loop gain L_i: total product of gains along a loop
- Loop i and loop j are non-touching if they do not share any nodes or branches

- The determinant Δ:

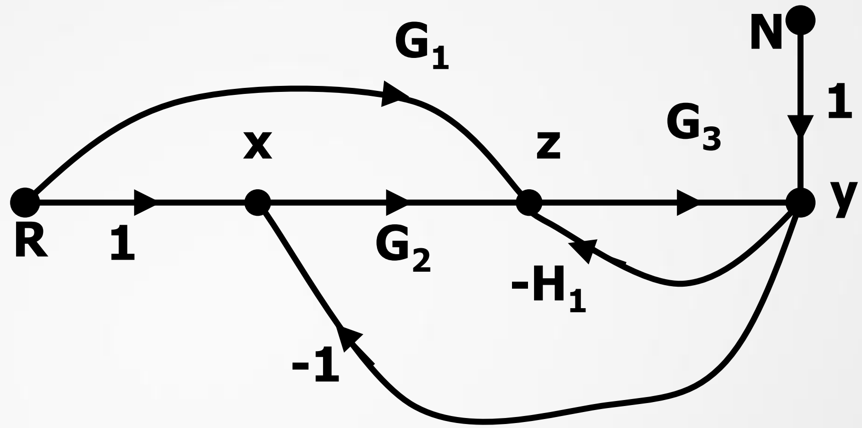
$$\Delta = 1 - \sum_{all\ i} L_i + \sum_{\substack{all\ non-touching \\ pairs\ of\ loops}} L_i \cdot L_j - \sum_{\substack{all\ n.t. \\ 3 loops}} L_i \cdot L_j \cdot L_k + \sum_{\substack{all\ n.t. \\ 4 loops}} L_i \cdot L_j \cdot L_k \cdot L_m - \ldots$$

- Δ_k: The determinant of the S.F.G. after removing the k-th forward path

- Mason's Gain formula:

$$I/O\ T.F. = \frac{y_o}{y_i} = \sum_{\substack{all\ forward \\ path\ k}} \frac{M_k \cdot \Delta_k}{\Delta}$$

Get T.F. from N to y



1 forward path: N → y

M = 1

2 loops: $L_1 = -H_1G_3$

$L_2 = -G_2G_3$

$$\Delta = 1 - \sum_{all} L_i + \cancelto{0}{\sum_{N.T.} L_i \cdot L_j} = 1 + G_3H_1 + G_2G_3$$

Δ_1: remove nodes N, y, and branch N → y

All loops broken: ➔ $\Delta_1 = 1$

$$\frac{y}{N} = \sum \frac{M_k \Delta_k}{\Delta} = \frac{M_1 \Delta_1}{\Delta} = \frac{1}{\Delta} = \frac{1}{1 + G_3H_1 + G_2G_3}$$

Get T.F. from R to y

2 f.p.: R x z y : $M_1 = G_2G_3$

R z y : $M_2 = G_1G_3$

2 loops: $L_1 = -G_3H_1$

$L_2 = -G_2G_3$

$$\Delta = 1 - \sum_{all} L_i + \sum_{N.T.} L_i \cdot L_j = 1 + G_3H_1 + G_2G_3$$

0

Δ_1: remove M_1 and compute Δ

$\Delta_1 = 1$

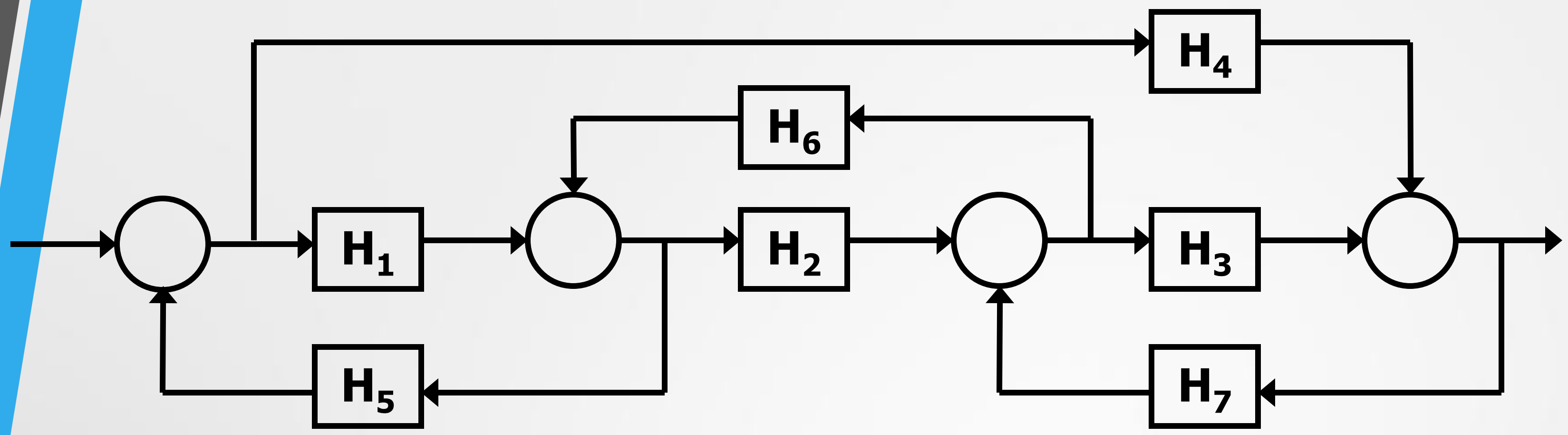
Δ_2: remove M_2 and compute Δ

$\Delta_2 = 1$

$$H = \frac{y}{R} = \sum \frac{M_k \Delta_k}{\Delta} = \sum \frac{M_k}{\Delta} = \frac{G_2 G_3 + G_1 G_3}{1 + G_3 H_1 + G_2 G_3}$$

Overall:

$$y = \frac{G_2 G_3 + G_1 G_3}{1 + G_3 H_1 + G_2 G_3} R + \frac{1}{1 + G_3 H_1 + G_2 G_3} N$$



Forward path:

$M_1 = H_1 H_2 H_3$

$M_2 = H_4$

Loops:

$L_1 = H_1 H_5$

$L_2 = H_2 H_6$

$L_3 = H_3 H_7$

$L_4 = H_4 H_7 H_6 H_5$

L_1 and L_3 are non-touching

$$\Delta = 1 - \sum L_i + L_1 L_3$$
$$= 1 - H_1 H_5 - H_2 H_6 - H_3 H_7 - H_4 H_7 H_6 H_5$$
$$+ H_1 H_5 H_3 H_7$$

Δ_1: If M_1 is taken out, all loops are broken.

therefore $\Delta_1 = 1$

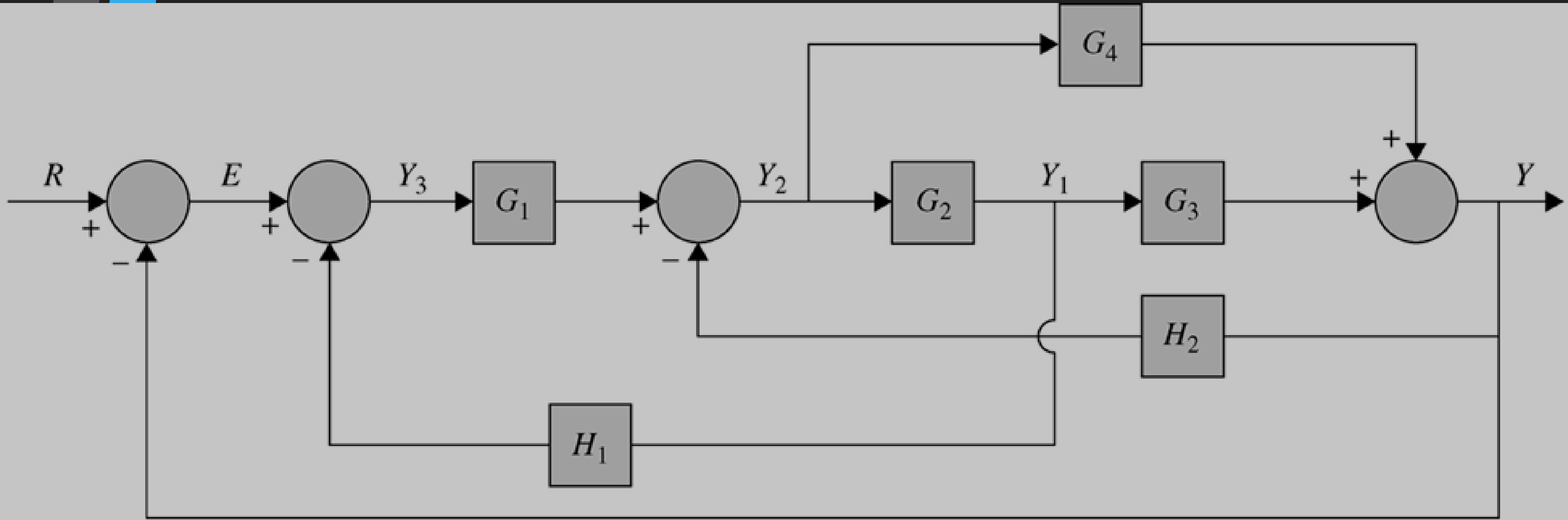
Δ_2: If M_2 is taken out, the loop in the middle (L_2) is still there.

therefore $\Delta_2 = 1 – L_2 = 1 – H_2H_6$

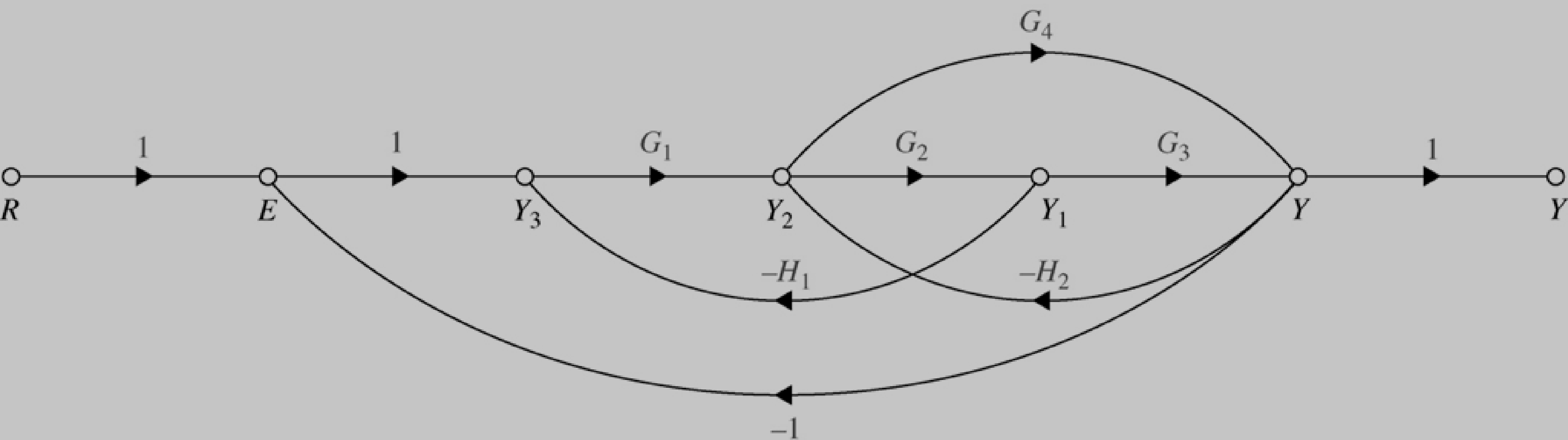
Total T.F.:

$$H = \sum \frac{M_k \Delta_k}{\Delta} = \frac{M_1 + M_2(1 - H_2 H_6)}{\Delta}$$

$$= \frac{H_1 H_2 H_3 + H_4 - H_4 H_2 H_6}{1 - H_1 H_5 - H_2 H_6 - H_3 H_7 - H_4 H_7 H_6 H_5 + H_1 H_5 H_3 H_7}$$



(a)



(b)

Two forward paths:

$$M_1 = G_1 G_2 G_3$$

$$M_2 = G_1 G_4$$

Five loops:

$$G_1 G_2 H_1 \qquad G_2 G_3 H_2 \qquad G_4 H_2$$

$$G_1 G_2 G_3 \qquad G_1 G_4$$

Determinant :

$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4$$

After removing forward path 1, no loops.

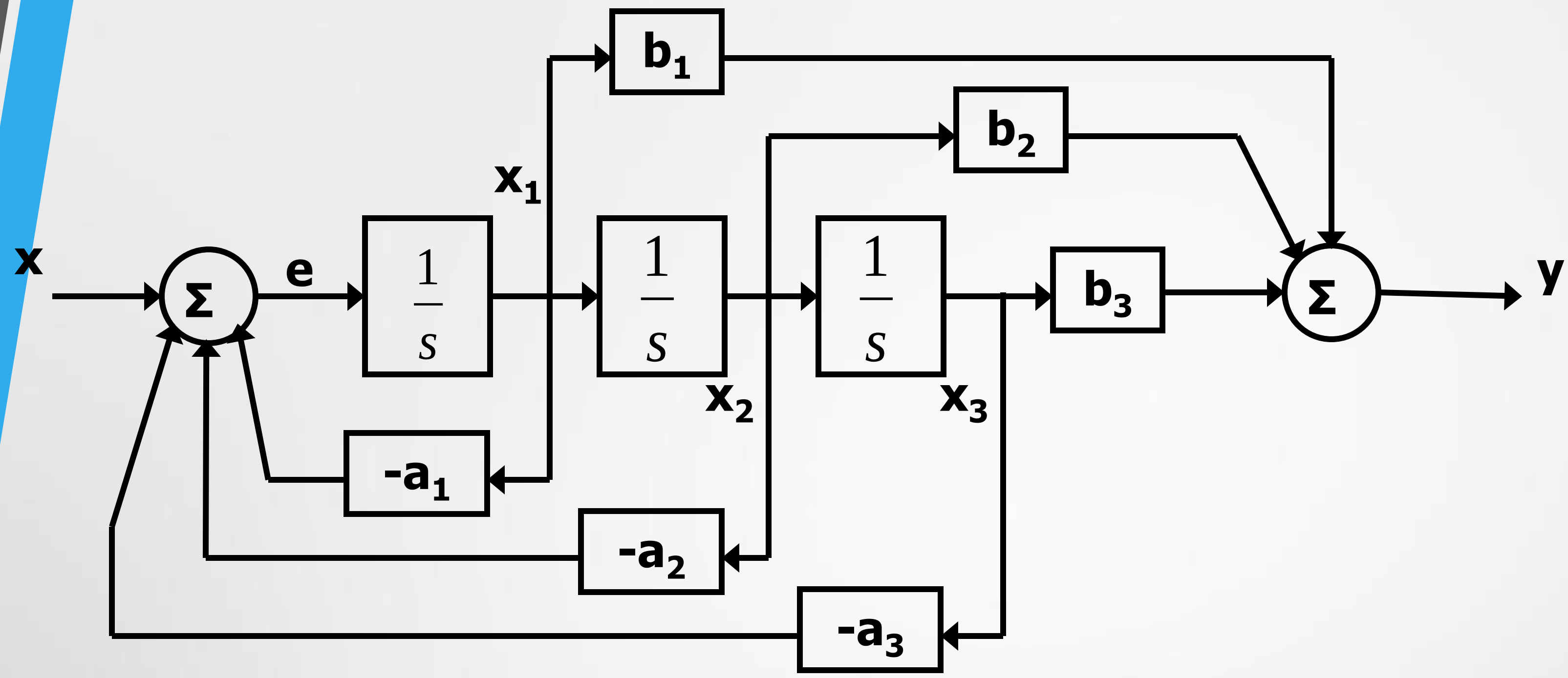
$$\therefore \quad \Delta_1 = 1$$

After removing forward path 2, no loops.

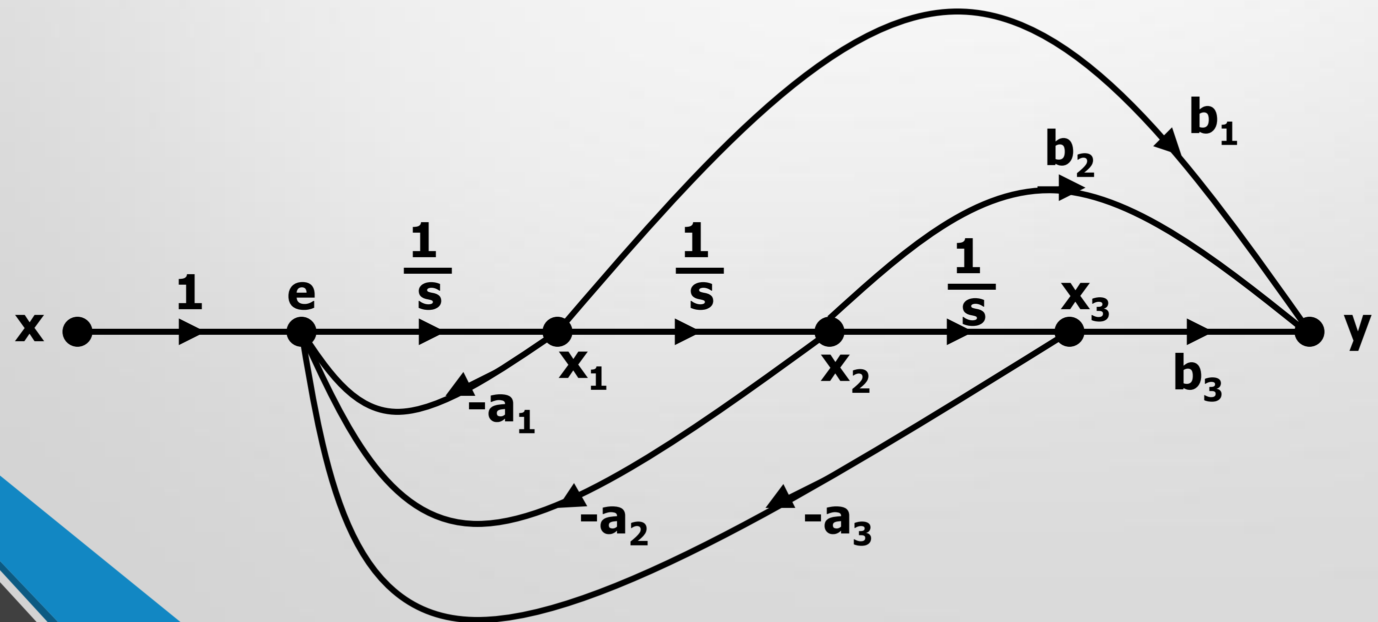
$$\therefore \quad \Delta_2 = 1$$

Total gain :

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}$$

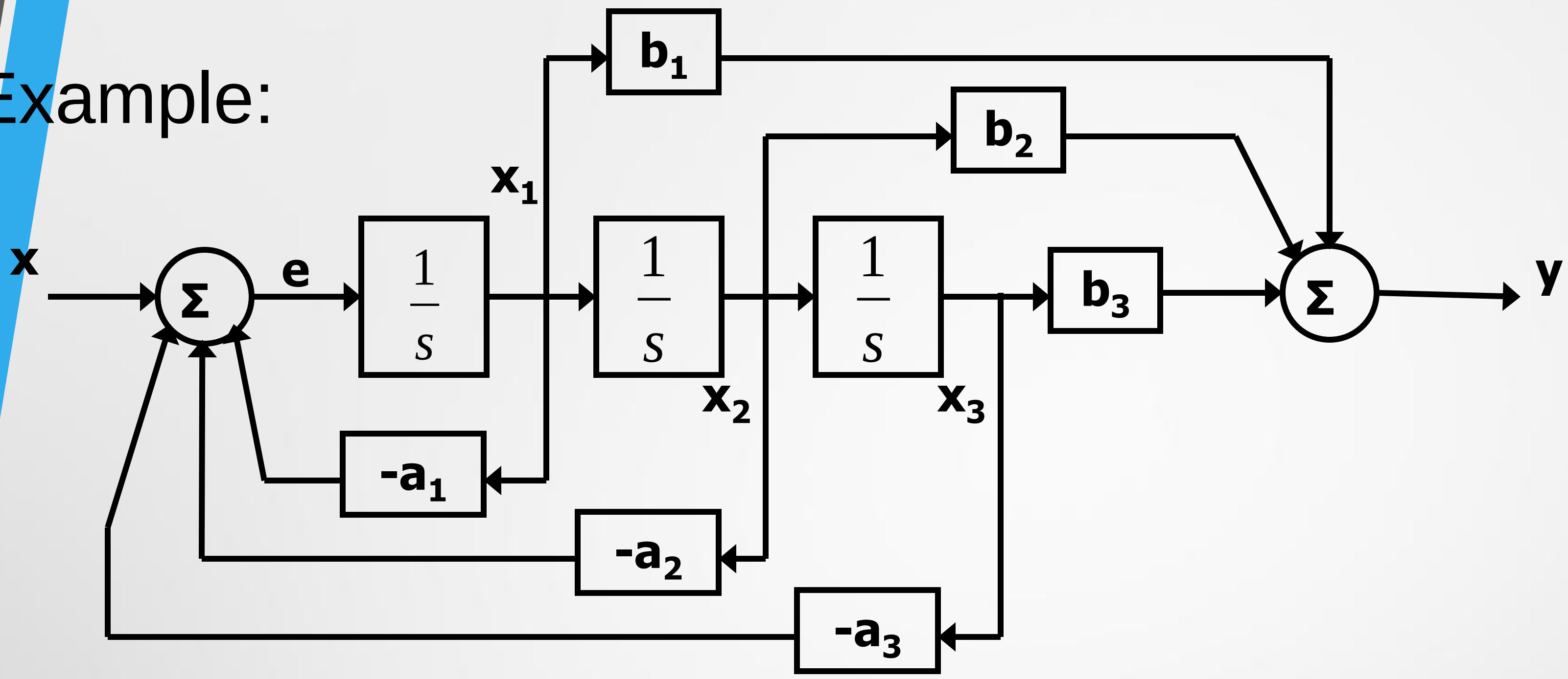


b_1
b_2
x_1
x
e
$\frac{1}{s}$
$\frac{1}{s}$
$\frac{1}{s}$
b_3
y
x_2
x_3
$-a_1$
$-a_2$
$-a_3$



b_1
b_2
$\frac{1}{s}$
$\frac{1}{s}$
$\frac{1}{s}$
x
1
e
x_3
y
x_1
x_2
b_3
$-a_1$
$-a_2$
$-a_3$

Example:



- Forward paths:

$$M_1 = \frac{b_3}{s^3}$$

$$M_2 = \frac{b_2}{s^2}$$

$$M_3 = \frac{b_1}{s}$$

- Loops:

$$L_1 = -\frac{a_1}{s}$$

$$L_2 = -\frac{a_2}{s^2}$$

$$L_3 = -\frac{a_3}{s^3}$$

Determinant:

$$\Delta = 1 - \sum_{all\ i} L_i = 1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}$$

Δ_1: If M_1 is taken out, all loops are broken.

therefore $\Delta_1 = 1$

Δ_2: If M_2 is taken out, all loops are broken.

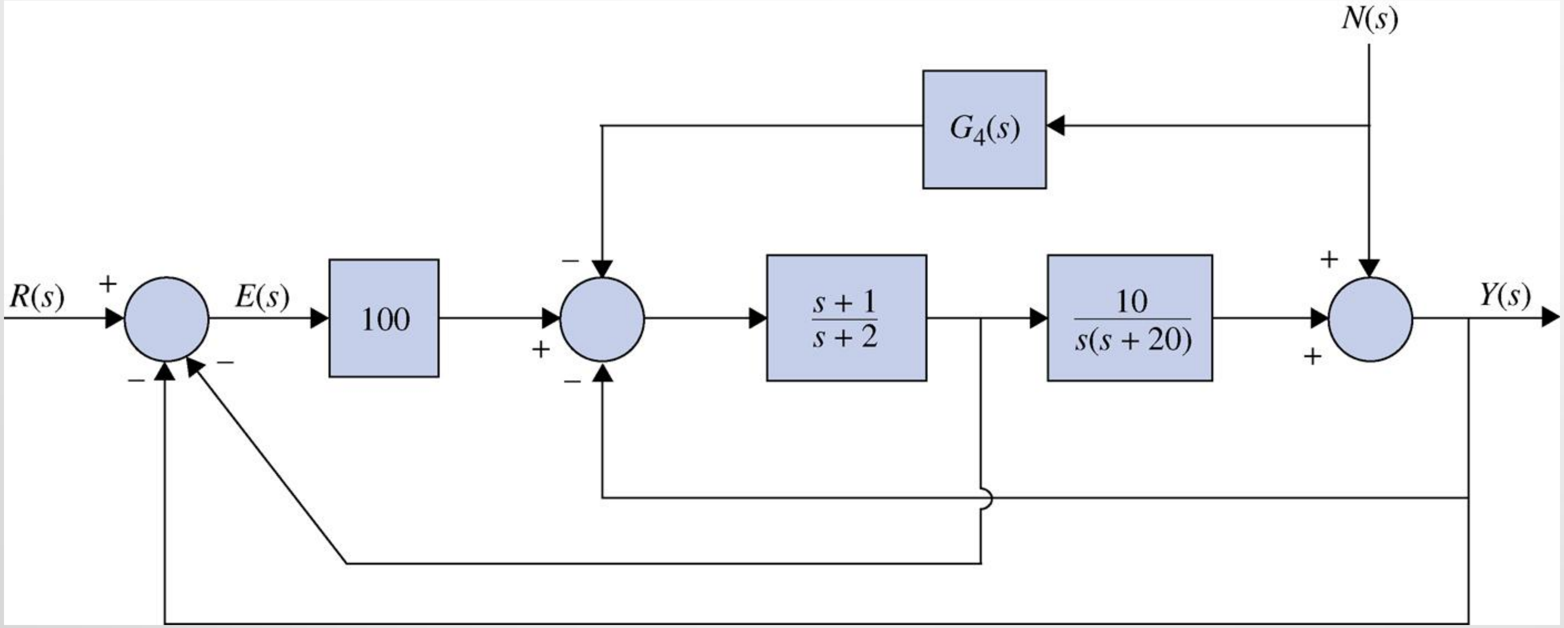
therefore $\Delta_2 = 1$

Δ_3: Similarly, $\Delta_3 = 1$

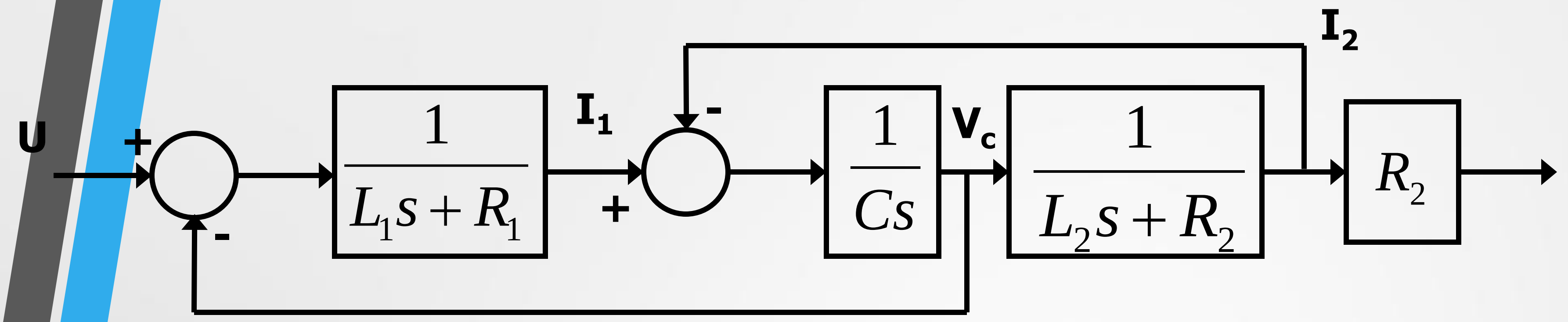
$$\therefore T.F. = \sum \frac{M_i \Delta_i}{\Delta} = \frac{M_1 M_2 M_3}{\Delta} = \frac{\frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}}$$

$$\Delta = 1 + 100\frac{s+1}{s+2}\frac{10}{s(s+20)} + 100\frac{s+1}{s+2} + \frac{s+1}{s+2}\frac{10}{s(s+20)}$$

$$= 1 + \frac{s+1}{s+2}\left(\frac{1000}{s(s+20)} + 1 + \frac{10}{s(s+20)}\right)$$



$$y = \frac{\frac{s+1}{s+2}\frac{1000}{s(s+20)}}{\Delta}R + \frac{\frac{s+1}{s+2}\frac{10}{s(s+20)}G_4 + 1*\left(1 + 100\frac{s+1}{s+2}\right)}{\Delta}N$$



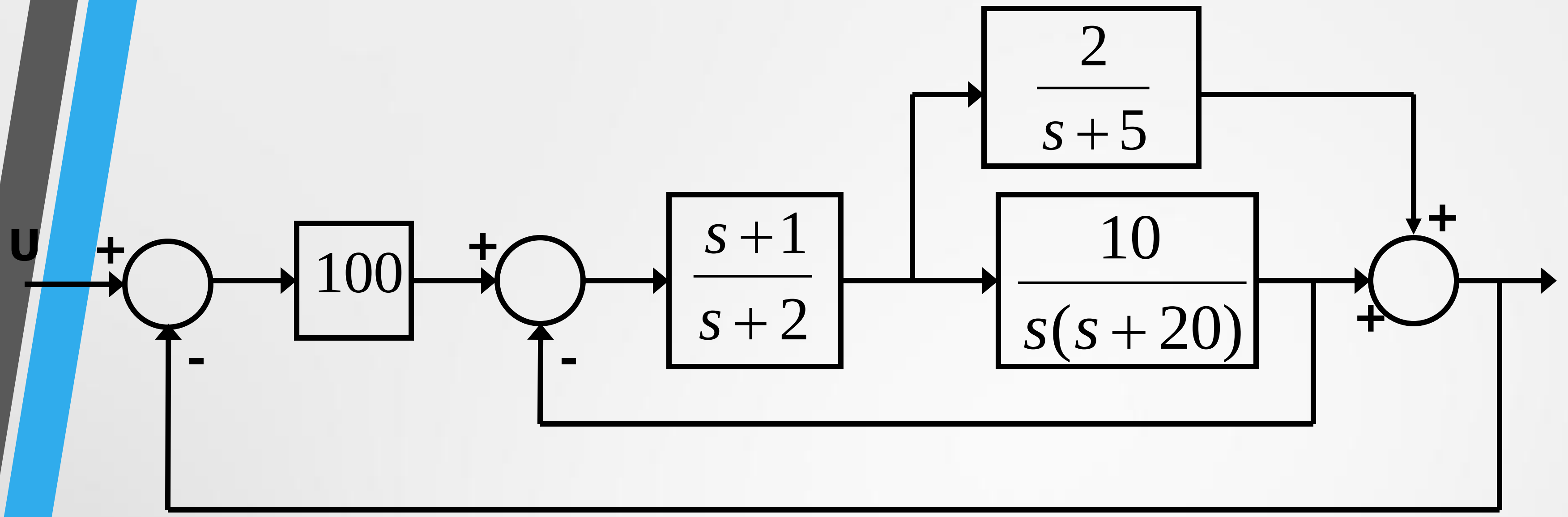
One forward path, two loops, no non-touching loops.

$$L1=-\frac{1}{L_1 s+R_1}\frac{1}{Cs}; \qquad L2=-\frac{1}{Cs}\frac{1}{L_2 s+R_2}$$

$$M1=\frac{1}{L_1 s+R_1}\frac{1}{Cs}\frac{1}{L_2 s+R_2}R_2; \qquad \Delta 1=1$$

$$T.F.=\frac{M1 \cdot \Delta 1}{1-L1-L2}$$

$$T.F.=\frac{R_2}{Cs(L_1 s+R_1)(L_2 s+R_2)+L_2 s+R_2+L_1 s+R_1}$$



Two forward paths, three loops, no non-touching loops.

$$M1 = 100\frac{s+1}{s+2}\frac{10}{s(s+20)}; \qquad \Delta 1 = 1$$

$$M1 = 100\frac{s+1}{s+2}\frac{2}{s+5}; \qquad \Delta 2 = 1$$

$$L1 = -100\frac{s+1}{s+2}\frac{10}{s(s+20)};$$

$$L2 = -\frac{s+1}{s+2}\frac{10}{s(s+20)}; \qquad L1 = -100\frac{s+1}{s+2}\frac{2}{s+5}$$